ANALYSIS 1 EXAMPLES SHEET 4

Lent Term 2014 W. T. G.

1. Show directly from the definition of an integral that $\int_0^a x^2 dx = a^3/3$ for $a > 0$.

2. Give an example of a continuous function $f : [0,\infty) \to [0,\infty)$ such that $\int_0^\infty f(x)dx$ exists but f is unbounded.

3. Give an example of an integrable function $f : [0,1] \to \mathbb{R}$ such that $f(x) \geq 0$ for every x, $f(y) > 0$ for some y, and $\int_0^1 f(x)dx = 0$.

Prove that this cannot happen if in addition f is continuous.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be monotonic. Show that the set of x such that f is discontinuous at x is countable.

Let (x_n) be a sequence of distinct points in $(0,1]$. Let $f_n(x) = 0$ if $0 \leq x < x_n$ and let $f_n(x) = 1$ if $x_n \leq x \leq 1$. For each x, let $f(x) = \sum_{n=1}^\infty 2^{-n} f_n(x)$. Prove that this series converges for every $x \in [0,1]$.

Explain why f must be integrable.

Prove that f is discontinuous at every x_n.

5. Define a function $f : [0,1] \to \mathbb{R}$ as follows. If x is irrational, then $f(x) = 0$. If x is rational, then write it in its lowest terms as p/q and then $f(x) = 1/q$. Prove that f is integrable. What is $\int_0^1 f(x)dx$?

6. Let $a < b$ and let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function such that $f(x) \geq 0$ for every x. Prove that if $\int_a^b f(x)dx = 0$, then for every closed subinterval $I \subset [a,b]$ of positive length and every $\epsilon > 0$ there exists a closed subinterval $J \subset I$ of positive length such that $f(x) \leq \epsilon$ for every $x \in J$.

Deduce that if $f(x) > 0$ for every x, then $\int_a^b f(x)dx > 0$.

7. Do these improper integrals converge?

(i) $\int_1^\infty \sin^2(1/x)dx$.

(ii) $\int_0^\infty x^p \exp(-x^q)dx$ (with $p, q > 0$).

(iii) $\int_0^\infty \sin(x^2)dx$.

8. Prove that $\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\to\log 2$ as $n\to\infty$, and find the limit of $\frac{1}{n+1}-\frac{1}{n+2}+\cdots+\frac{(-1)^{n-1}}{2n}$.

9. Let $f:[a,b]\to\mathbb{R}$ be continuous and suppose that $\int_a^b f(x)g(x)dx=0$ for every continuous function $g:[a,b]\to\mathbb{R}$ with $g(a)=g(b)=0$. Must f vanish identically?

10. Let $f:[0,1]\to\mathbb{R}$ be continuous. Let $G(x,t)=t(x-1)$ when $t\leq x$ and $x(t-1)$ when $t\geq x$. Let $g(x)=\int_0^1 f(t)G(x,t)dt$. Show that $g''(x)$ exists for $x\in(0,1)$ and equals $f(x)$.

11. For positive x, define $L(x)$ to be $\int_1^x \frac{dt}{t}$. Prove directly from this definition that the function L has the properties one normally expects of the logarithm function. In particular, prove that $L(ab)=L(a)+L(b)$ for all positive a and b. If you adopted this as your fundamental definition of natural logarithms, then how would you define e?

12. For each non-negative integer n let $I_n(\theta)=\int_{-1}^1(1-x^2)^n\cos(\theta x)dx$. Prove that $\theta^2 I_n = 2n(2n-1)I_{n-1}-4n(n-1)I_{n-2}$ for all $n\geq 2$, and hence that $\theta^{2n+1}I_n(\theta)=n!(P_n(\theta)\sin\theta+Q_n(\theta)\cos\theta)$ for some pair P_n and Q_n of polynomials of degree at most $2n$ with integer coefficients.

Deduce that π is irrational.

13. Let $f:[-1,1]$ be defined by $f(x)=x\sin(1/x)$ when $x\neq 0$ and $f(0)=0$. Explain why f is integrable. Prove that there do not exist increasing functions g and h, defined on $[-1,1]$, such that $f(x)=g(x)-h(x)$ for every x.

14. Prove that if $f:[0,1]\to\mathbb{R}$ is integrable, then f has infinitely many points of continuity.

15*. Let $f:[0,1]\to\mathbb{R}$ be a function that is differentiable everywhere (with right and left derivatives at the end points) with a derivative f' that is bounded. Must f' be integrable?

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