ANALYSIS 1 EXAMPLES SHEET 3

Lent Term 2014 W. T. G.

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq |x - y|^2$ for every $x, y \in \mathbb{R}$. Prove that f is constant.

2. (i) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$. Prove that f is differentiable everywhere. For which x is f' continuous at x?

(ii) Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ that is differentiable everywhere such that g' is not bounded on the interval $[-1, 1]$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with the property that $f(x) = o(x^n)$ for every positive integer n. (In other words, for every n we have $f(x)/x^n \to 0$ as $x \to 0$.) Does it follow that f is infinitely differentiable at 0?

4. By applying the mean value theorem to $\log(1 + x)$ on the interval $[0, a/n]$, prove rigorously that $(1 + a/n)^n \to e^a$ as $n \to \infty$.

5. Find $\lim_{n\to\infty} n(a^{1/n} - 1)$, when $a > 0$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \exp(-1/x^2)$ when $x \neq 0$ and $f(0) = 0$. Prove that f is infinitely differentiable and that $f^{(n)}(0) = 0$ for every $n \in \mathbb{N}$. What does Taylor's theorem tell us when we apply it to f at 0?

7. Find the radius of convergence of each of the following power series.

$$\sum_{n=0}^{\infty} \frac{2.4.6\ldots(2n+2)}{1.4.7\ldots(3n+1)} z^n \qquad \sum_{n=1}^{\infty} \frac{z^{3n}}{n2^n} \qquad \sum_{n=0}^{\infty} \frac{n^n z^n}{n!} \qquad \sum_{n=1}^{\infty} n^{\sqrt{n}} z^n$$

8. Find the derivative of $\tan x$ on the interval $(-\pi/2, \pi/2)$. How do you know that there is a differentiable inverse function $\arctan x$ from $\mathbb{R}$ to $(-\pi/2, \pi/2)$? What is its derivative? By considering derivatives, prove that $\arctan x = x - x^3/3 + x^5/5 - \ldots$ when $|x| < 1$.

9. Let f and g be two functions defined and differentiable on an open interval I containing 0. Suppose that $f(0) = g(0) = 0$ and that $f'(x)/g'(x)$ converges to a limit ℓ as $x \to 0$.

(i) Show that there is an open interval of the form $(0, a)$ on which g' does not vanish. Let $0 < x < a$. By considering the function $F(u) = f(x)g(u) - g(x)f(u)$, prove that there exists y with $0 < y < x$ such that $\frac{f'(y)}{g'(y)} = \frac{f(x)}{g(x)}$. Explain briefly why a similar statement holds for negative x.

(ii) Deduce *l'Hôpital's rule*, which states that under the conditions above, $f(x)/g(x) \to \ell$.

(iii) What is $\lim_{x\to 0}(1-\cos(\sin x))/x^2$?

10. Let (a_n) be a bounded real sequence. Prove that (a_n) has a subsequence that tends to $\limsup a_n$. What result from the course does this imply?

11. The infinite product $\prod_{n=1}^{\infty}(1+a_n)$ is said to converge to a if the sequence of partial products $P_n = (1+a_1)\dots(1+a_n)$ converges to a. Suppose that $a_n \geq 0$ for every n. Write $S_n = a_1 + \dots + a_n$. Prove that $S_n \leq P_n \leq e^{S_n}$ for every n, and deduce that $\prod_{n=1}^{\infty}(1+a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. Evaluate the product $\prod_{n=2}^{\infty}(1+1/(n^2-1))$.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, let a and b be real numbers with $a < b$, and suppose that $f'(a) < 0 < f'(b)$. Prove that there exists $c \in (a,b)$ such that $f'(c) = 0$. Deduce the more general result that if $f'(a) \neq f'(b)$ and z lies between $f'(a)$ and $f'(b)$, then there exists $c \in (a,b)$ such that $f'(c) = z$. (This result is called *Darboux's theorem*.)

13. Say that an ordered field $\mathbb{F}$ has the *intermediate value property* if for every $a < b$ and every continuous function $f : \mathbb{F} \to \mathbb{F}$, if $f(a) < 0$ and $f(b) > 0$ then there exists $c \in (a,b)$ such that $f(c) = 0$. Prove that every ordered field with the intermediate value property has the least upper bound property. (This implies that it is isomorphic to $\mathbb{R}$.)

14. (i) Show that the series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ has radius of convergence 1, and that it converges for every z such that $|z| = 1$, with the exception of $z = 1$.

(ii) Let $z_1, \dots, z_m$ be complex numbers of modulus 1. Find a power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence 1 that converges for every z such that $|z| = 1$, except when $z \in \{z_1, \dots, z_m\}$, when it diverges.

15. (i) Let f and g be two n-times-differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. For $k \leq n$ and $x \in \mathbb{R}$, say that f and g *agree to order k at x* if if $f^{(j)}(x) = g^{(j)}(x)$ for $j = 0, 1, \dots, k-1$. Let $x_1 < x_2 < \dots < x_r$ be real numbers, let $k_1, \dots, k_r$ be non-negative integers such that $k_1 + \dots + k_r = n$, and suppose that for each $i \leq r$ the functions f and g agree to order k_i at x_i. If $r \geq 2$, prove that there exists x in the open interval (x_1, x_r) such that $f^{(n-1)}(x) = g^{(n-1)}(x)$. [Note that if you can do this when g is the zero function then you can do it in general. If you still find it too hard, then try it in the case $r = n$, so $k_1 = \dots = k_n = 1$, and in the case $k = 2$, to get an idea what is going on.]

(ii) Let f be n-times differentiable, let $x_1 < \dots < x_r$ be real numbers and let $k_1, \dots, k_r$ be non-negative integers with $k_1 + \dots + k_r = n$. Prove that there is a polynomial p of degree at most $n-1$ such that for every $i \leq r$ and every $j < k_i$ we have $p^{(j)}(x_i) = f^{(j)}(x_i)$. [Hint: start by building a suitable basis of polynomials and then take linear combinations.]

(iii) Write down an expression for the constant value of $p^{(n-1)}$.