

# ANALYSIS I EXAMPLES 3

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at [g.p.paternain@dpmms.cam.ac.uk](mailto:g.p.paternain@dpmms.cam.ac.uk).

1. Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfies  $|f(x) - f(y)| \leq |x - y|^2$  for all  $x, y \in \mathbb{R}$ . Show that  $f$  is constant.
2. Given  $\alpha \in \mathbb{R}$ , define  $f_\alpha : [-1, 1] \rightarrow \mathbb{R}$  by  $f_\alpha(x) = x^\alpha \sin(1/x)$  for  $x \neq 0$  and  $f_\alpha(0) = 0$ . Is  $f_0$  continuous? Is  $f_1$  differentiable? Draw a table, with 4 columns labelled 0, 1, 2, 3 and with 6 rows labelled “ $f_\alpha$  bounded”, “ $f_\alpha$  continuous”, “ $f_\alpha$  differentiable”, “ $f'_\alpha$  bounded”, “ $f'_\alpha$  continuous”, “ $f'_\alpha$  differentiable”. Place ticks and crosses at appropriate places in the table.  
Does  $|x|^\alpha \sin(1/x)$  behave the same way? Complete 5 extra columns, for  $\alpha = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$ .
3. By applying the mean value theorem to  $\log(1 + x)$  on  $[0, a/n]$  with  $n > |a|$ , prove carefully that  $(1 + a/n)^n \rightarrow e^a$  as  $n \rightarrow \infty$ .
4. Find  $\lim_{n \rightarrow \infty} n(a^{1/n} - 1)$ , where  $a > 0$ .
5. “Let  $f'$  exist on  $(a, b)$  and let  $c \in (a, b)$ . If  $c + h \in (a, b)$  then  $(f(c + h) - f(c))/h = f'(c + \theta h)$ . Let  $h \rightarrow 0$ ; then  $f'(c + \theta h) \rightarrow f'(c)$ . Thus  $f'$  is continuous at  $c$ .” Is this argument correct?
6. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = \exp(-1/x^2)$  for  $x \neq 0$  and  $f(0) = 0$ . Show that  $f$  is continuous and differentiable. Show that  $f$  is twice differentiable. Indeed, show that  $f$  is infinitely differentiable, and that  $f^{(n)}(0) = 0$  for all  $n \in \mathbb{N}$ . Comment, in the light of what you know about Taylor series.
7. Find the radius of convergence of each of these power series.

$$\sum_{n \geq 0} \frac{2 \cdot 4 \cdot 6 \cdots (2n+2)}{1 \cdot 4 \cdot 7 \cdots (3n+1)} z^n \qquad \sum_{n \geq 1} \frac{z^{3n}}{n2^n} \qquad \sum_{n \geq 0} \frac{n^n z^n}{n!} \qquad \sum_{n \geq 1} n^{\sqrt{n}} z^n$$

8. (L'Hôpital's rule.) Suppose that  $f, g : [a, b] \rightarrow \mathbb{R}$  are continuous and differentiable on  $(a, b)$ . Suppose that  $f(a) = g(a) = 0$ , that  $g'(x)$  does not vanish near  $a$  and  $f'(x)/g'(x) \rightarrow \ell$  as  $x \rightarrow a$ . Show that  $f(x)/g(x) \rightarrow \ell$  as  $x \rightarrow a$ . Use the rule with  $g(x) = x - a$  to show that if  $f'(x) \rightarrow \ell$  as  $x \rightarrow a$ , then  $f$  is differentiable at  $a$  with  $f'(a) = \ell$ .

Find a pair of functions  $f$  and  $g$  as above for which  $\lim_{x \rightarrow a} f(x)/g(x)$  exists, but  $\lim_{x \rightarrow a} f'(x)/g'(x)$  does not.

Investigate the limit as  $x \rightarrow 1$  of

$$\frac{x - (n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}.$$

9. Find the derivative of  $\tan x$ . How do you know there is a differentiable inverse function  $\tan^{-1} x$  for  $x \in \mathbb{R}$ ? What is its derivative? Now let  $g(x) = x - x^3/3 + x^5/5 + \cdots$  for  $|x| < 1$ . By considering  $g'(x)$ , explain carefully why  $\tan^{-1} x = g(x)$  for  $|x| < 1$ .
10. The *infinite product*  $\prod_{n=1}^{\infty} (1 + a_n)$  is said to *converge* if the sequence  $p_n = (1 + a_1) \cdots (1 + a_n)$  converges. Suppose that  $a_n \geq 0$  for all  $n$ . Putting  $s_m = a_1 + \cdots + a_m$ , prove that  $s_n \leq p_n \leq e^{s_n}$ , and deduce that  $\prod_{n=1}^{\infty} (1 + a_n)$  converges if and only if  $\sum_{n=1}^{\infty} a_n$  converges. Evaluate  $\prod_{n=2}^{\infty} (1 + 1/(n^2 - 1))$ .
11. Let  $f$  be continuous on  $[-1, 1]$  and twice differentiable on  $(-1, 1)$ . Let  $\phi(x) = (f(x) - f(0))/x$  for  $x \neq 0$  and  $\phi(0) = f'(0)$ . Show that  $\phi$  is continuous on  $[-1, 1]$  and differentiable on  $(-1, 1)$ . Using

a second order mean value theorem for  $f$ , show that  $\phi'(x) = f''(\theta x)/2$  for some  $0 < \theta < 1$ . Hence prove that there exists  $c \in (-1, 1)$  with  $f''(c) = f(-1) + f(1) - 2f(0)$ .

**12.** Prove the theorem of Darboux: that if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is differentiable then  $f'$  has the “property of Darboux”. (That is to say, if  $a < b$  and  $f'(a) < z < f'(b)$  then there exists  $c$ ,  $a < c < b$ , with  $f'(c) = z$ .)

**13.** Using Question 6, construct a function  $g : \mathbb{R} \rightarrow \mathbb{R}$  that is infinitely-differentiable, positive on a given interval  $(a, b)$  and zero elsewhere. Now set

$$f(x) = \frac{\int_{-\infty}^x g}{\int_{-\infty}^{\infty} g}.$$

Show that  $f$  is infinitely-differentiable,  $f(x) = 0$  for  $x < a$ ,  $f(x) = 1$  for  $x > b$  and  $0 < f(x) < 1$  for  $x \in (a, b)$ . [For this part of the question you may assume standard properties of integration, including that  $f'(x) = g(x)/\int_{-\infty}^{\infty} g$ .]

Finally, construct a function from  $\mathbb{R}$  to  $\mathbb{R}$  that is infinitely-differentiable, but is identically 1 on  $[-1, 1]$  and identically 0 outside  $(-2, 2)$ .