Analysis 1 (2008–09) Example Sheet 4 of 4

1. Show directly from the definition of an integral that $\int_0^a x^2 = a^3/3$ for $a > 0$.

2. Let $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Does $\int_0^1 f$ exist?

3. Give an example of continuous function $f : [0, \infty) \to [0, \infty)$, such that $\int_0^\infty f$ exists but f is unbounded.

4. Give an example of an integrable function $f : [0, 1] \to \mathbb{R}$ with $f \geq 0$, $\int_0^1 f = 0$, and $f(x) > 0$ for some value of x. Show that this cannot happen if f is continuous.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be monotonic. Show that $\{x \in \mathbb{R} : f \text{ is discontinuous at } x\}$ is countable.
Let x_n, $n \geq 1$ be a sequence of distinct points in $(0, 1]$. Let $f_n(x) = 0$ if $0 \leq x < x_n$ and $f_n(x) = 1$ if $x_n \leq x \leq 1$. Let $f(x) = \sum_{n=1}^\infty 2^{-n} f_n(x)$. Show that this series converges for every $x \in [0, 1]$. Show that f is increasing (and so is integrable). Show that f is discontinuous at every x_n.

6. Let $f(x) = \log(1 - x^2)$. Use the mean value theorem to show that $|f(x)| \leq 8x^2/3$ for $0 \leq x \leq 1/2$. Now let $I_n = \int_{n-1/2}^{n+1/2} \log x \, dx - \log n$ for $n \in \mathbb{N}$. Show that $I_n = \int_0^{1/2} f(t/n) \, dt$ and hence that $|I_n| < 1/9n^2$. By considering $\sum_{j=1}^n I_j$, deduce that $n!/n^{n+1/2}e^{-n} \to \ell$ for some constant ℓ.
[The bounds $8x^2/3$ and $1/9n^2$ are not best possible; they are merely good enough for the conclusion.]

7. Let $I_n = \int_0^{\pi/2} \cos^n x$. Prove that $nI_n = (n - 1)I_{n-2}$, and hence $\frac{2n}{2n+1} \leq I_{2n+1}/I_{2n} \leq 1$.

Deduce Wallis's Product:

$$\frac{\pi}{2} = \lim_{n\to\infty} \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdots 2n \cdot 2n}{1 \cdot 3 \cdot 3 \cdot 5 \cdots (2n-1) \cdot (2n+1)} = \lim_{n\to\infty} \frac{2^{4n}}{2n+1} \binom{2n}{n}^{-2}.$$

By taking note of the previous exercise, prove that $n!/n^{n+1/2}e^{-n} \to \sqrt{2\pi}$ (Stirling's formula).

8. Do these improper integrals converge? (i) $\int_1^\infty \sin^2(1/x)dx$, (ii) $\int_0^\infty x^p \exp(-x^q)dx$ where $p, q > 0$.

9. Show that $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \to \log 2$ as $n \to \infty$, and find $\lim_{n\to\infty} \frac{1}{n+1} - \frac{1}{n+2} + \cdots + \frac{(-1)^{n-1}}{2n}$.

10. Suppose that $f : \mathbb{R} \to \mathbb{R}$ has a continuous derivative, $f(0) = 0$ and $|f'(x)| \leq M$ for $x \in [0, 1]$. Show that $|\int_0^1 f| \leq M/2$. Show that if, in addition, $f(1) = 0$ then $|\int_0^1 f| \leq M/4$. What could you say if $|f'(x)| \leq Mx$?

11. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Let $G(x, t) = t(x - 1)$ for $t \leq x$ and $G(x, t) = x(t - 1)$ for $t \geq x$. Let $g(x) = \int_0^1 f(t)G(x, t)dt$. Show that $g''(x)$ exists for $x \in (0, 1)$ and equals $f(x)$.

12. Let $I_n(\theta) = \int_{-1}^1 (1 - x^2)^n \cos(\theta x)dx$. Prove that $\theta^2 I_n = 2n(2n-1)I_{n-1} - 4n(n-1)I_{n-2}$ for $n \geq 2$, and hence that $\theta^{2n+1}I_n(\theta) = n!(P_n(\theta)\sin\theta + Q_n(\theta)\cos\theta)$, where P_n and Q_n are polynomials of degree at most $2n$ with integer coefficients. Deduce that π is irrational.

13. Let $f_1, f_2 : [-1, 1] \to \mathbb{R}$ be increasing and $g = f_1 - f_2$. Show that there exists K such that, for any dissection $\mathcal{D} = x_0 < \ldots < x_n$ of $[-1, 1]$, $\sum_{j=1}^n |g(x_j) - g(x_{j-1})| \leq K$. Now let $g(x) = x\sin(1/x)$ for $x \neq 0$ and $g(0) = 0$. Show that g is integrable but is not the difference of two increasing functions.

14. Give an example of functions $f : [0, 1] \to [0, 1]$ and $g : [0, 1] \to [0, 1]$ which are both integrable but such that $f \circ g$ is not integrable. (The notation means $(f \circ g)(x) = f(g(x))$.) Show that there is no example with f continuous. $^+$Is there an example with g continuous?

15. Show that if $f : [0, 1] \to \mathbb{R}$ is integrable then f has a point of continuity.