1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq |x - y|^2$ for all $x, y \in \mathbb{R}$. Show that f is constant.

2. Given $\alpha \in \mathbb{R}$, define $f_\alpha : [-1, 1] \to \mathbb{R}$ by $f_\alpha(x) = x^\alpha \sin(1/x)$ for $x \neq 0$ and $f_\alpha(0) = 0$.
Is f_0 continuous? Is f_1 differentiable? Draw a table, with 4 columns labelled $0, 1, 2, 3$ and with 6 rows labelled "f_α bounded", "f_α continuous", "f_α differentiable", "f'_α bounded", "f'_α continuous", "f'_α differentiable". Place ticks and crosses at appropriate places in the table.

Does $|x|^\alpha \sin(1/x)$ behave the same way? Complete 5 extra columns, for $\alpha = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$,

3. By applying the mean value theorem to $\log(1 + x)$ on $[0, a/n]$ with $n > |a|$, prove carefully that $(1 + a/n)^n \to e^a$ as $n \to \infty$.

4. Find $\lim_{n\to\infty} n(a^{1/n} - 1)$, where $a > 0$.

5. "Let f' exist on (a, b) and let $c \in (a, b)$. If $c + h \in (a, b)$ then $(f(c + h) - f(c))/h = f'(c + \theta h)$. Let $h \to 0$; then $f'(c + \theta h) \to f'(c)$. Thus f' is continuous at c." Is this argument correct?

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \exp(-1/x^2)$ for $x \neq 0$ and $f(0) = 0$. Show that f is continuous and differentiable. Show that f is twice differentiable. Indeed, show that f is infinitely differentiable, and that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Comment, in the light of what you know about Taylor series.

7. Let $f(x) = x^{1/2}$. Express $f(1 + h)$ as a quadratic in h plus a remainder term involving h^3. By taking $h = -0.02$, find an approximate value for $\sqrt{2}$ and prove it is accurate to seven decimal places.

8. Find the radius of convergence of each of these power series.

$$\sum_{n\geq 0} \frac{2 \cdot 4 \cdot 6 \cdots (2n + 2)}{1 \cdot 4 \cdot 7 \cdots (3n + 1)} z^n \qquad \sum_{n\geq 1} \frac{z^{3n}}{n 2^n} \qquad \sum_{n\geq 0} \frac{n^n z^n}{n!} \qquad \sum_{n\geq 1} n^{\sqrt{n}} z^n$$

9. Find the derivative of $\tan x$. How do you know there is a differentiable inverse function $\tan^{-1} x$ for $x \in \mathbb{R}$? What is its derivative? Now let $g(x) = x - x^3/3 + x^5/5 + \cdots$ for $|x| < 1$. By considering $g'(x)$, explain carefully why $\tan^{-1} x = g(x)$ for $|x| < 1$.

If you so wish, verify Machin's formula $\frac{\pi}{4} = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}$ and thereby estimate π.

10. The *infinite product* $\prod_{n=1}^\infty (1 + a_n)$ is said to *converge* if the sequence $p_n = (1 + a_1) \cdots (1 + a_n)$ converges. Suppose that $a_n \geq 0$ for all n. Putting $s_m = a_1 + \cdots + a_m$, prove that $s_n \leq p_n \leq e^{s_n}$, and deduce that $\prod_{n=1}^\infty (1 + a_n)$ converges if and only if $\sum_{n=1}^\infty a_n$ converges. Evaluate $\prod_{n=2}^\infty (1 + 1/(n^2 - 1))$.

11. Let f be continuous on $[-1, 1]$ and twice differentiable on $(-1, 1)$. Let $\phi(x) = (f(x) - f(0))/x$ for $x \neq 0$ and $\phi(0) = f'(0)$. Using a second order mean value theorem for f, show that $\phi'(x) = f''(\theta x)/2$ for some $0 < \theta < 1$. Hence prove that there exists $c \in (-1, 1)$ with $f''(c) = f(-1) + f(1) - 2f(0)$.

12. Prove the theorem of Darboux: that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable then f' has the "property of Darboux". (That is to say, if $a < b$ and $f'(a) < z < f'(b)$ then there exists c, $a < c < b$, with $f'(c) = z$.)

13. Construct a function from $\mathbb{R}$ to $\mathbb{R}$ that is infinitely-differentiable, but is identically 1 on $[-1, 1]$ and identically 0 outside $(-2, 2)$.