Analysis I (2008–09) Example Sheet 2 of 4

1. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = 1 - x$ otherwise. Find $\{a : f \text{ is continuous at } a\}$.

2. Write down the definition of "$f(x) \to \infty$ as $x \to \infty$". Prove that $f(x) \to \infty$ as $x \to \infty$ if, and only if, $f(x_n) \to \infty$ for every sequence such that $x_n \to \infty$.

3. Suppose that $f(x) \to \ell$ as $x \to a$ and $g(y) \to k$ as $y \to \ell$. Must it be true that $g(f(x)) \to k$ as $x \to a$?

4. Let $f_n : [0,1] \to [0,1]$ be continuous, $n \in \mathbb{N}$. Let $h_n(x) = \max\{f_1(x), f_2(x), \ldots, f_n(x)\}$. Show that h_n is continuous on $[0,1]$ for each $n \in \mathbb{N}$. Must $h(x) = \sup\{f_n(x) : n \in \mathbb{N}\}$ be continuous?

5. The unit circle in $\mathbb{C}$ is mapped to $\mathbb{R}$ by a map $e^{i\theta} \mapsto f(\theta)$, where $f[0, 2\pi] \to \mathbb{R}$ is continuous and $f(0) = f(2\pi)$. Show that there exist two diametrically opposite points that have the same image.

6. Let $f(x) = \sin^2 x + \sin^2(x + \cos^7 x)$. Assuming the familiar features of $\sin$ without justification, prove that there exists $k > 0$ such that $f(x) \geq k$ for all $x \in \mathbb{R}$.

7. Suppose that $f : [0,1] \to \mathbb{R}$ is continuous, that $f(0) = f(1) = 0$, and that for every $x \in (0,1)$ there exists $0 < \delta < \min\{x, 1-x\}$ with $f(x) = (f(x-\delta) + f(x+\delta))/2$. Show that $f(x) = 0$ for all x.

8. Let $f : [a,b] \to \mathbb{R}$ be bounded. Suppose that $f((x+y)/2) \leq (f(x) + f(y))/2$ for all $x, y \in [a,b]$. Prove that f is continuous on (a,b). Must it be continuous at a and b too?

9. Prove that $2x^5 + 3x^4 + 2x + 16 = 0$ has no real solutions outside $[-2,-1]$ and exactly one inside.

10. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Which of (i)–(iv) must be true?

(i) If f is increasing then $f'(x) \geq 0$ for all $x \in (a,b)$.

(ii) If $f'(x) \geq 0$ for all $x \in (a,b)$ then f is increasing.

(iii) If f is strictly increasing then $f'(x) > 0$ for all $x \in (a,b)$.

(iv) If $f'(x) > 0$ for all $x \in (a,b)$ then f is strictly increasing.

[*Increasing* means $f(x) \leq f(y)$ if $x < y$, and *strictly increasing* means $f(x) < f(y)$ if $x < y$.]

11. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable for all x. Prove that if $f'(x) \to \ell$ as $x \to \infty$ then $f(x)/x \to \ell$. If $f(x)/x \to \ell$ as $x \to \infty$, must $f'(x)$ tend to a limit?

12. Let $f(x) = x + 2x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable everywhere and that $f'(0) = 1$, but that there is no interval around 0 on which f is increasing.

13. Define $f_0 : \mathbb{R} \to \mathbb{R}$ by $f_0(x) = |x|$ for $|x| \leq 1/2$ and $f_0(x+m) = f_0(x)$ for $m \in \mathbb{Z}$. Draw f_0. Now define $f_n : \mathbb{R} \to \mathbb{R}$ for $n \in \mathbb{N}$ by $f_n(x) = 4^{-n} f_0(4^n x)$. Draw f_1 and $f_0 + f_1$. Show that, for each $x \in \mathbb{R}$, $\sum_{n \geq 0} f_n(x)$ converges. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sum_{n \geq 0} f_n(x)$. Show that f is continuous everywhere. Where is f differentiable?

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