1. Prove that if $a_n \to a$ and $b_n \to b$ then $a_n + b_n \to a + b$.

2. Sketch the graphs of $y = x$ and $y = (x^4+1)/3$, and thereby illustrate the behaviour of the real sequence (a_n) where $a_{n+1} = (a_n^4+1)/3$. For which of the three starting cases $a_1 = 0$, $a_1 = 1$ and $a_1 = 2$ does the sequence converge? Now prove your assertion.

3. Let $a_1 > b_1 > 0$ and let $a_{n+1} = (a_n + b_n)/2$, $b_{n+1} = 2a_nb_n/(a_n+b_n)$ for $n \geq 1$. Show that $a_n > a_{n+1} > b_{n+1} > b_n$ and deduce that the two sequences converge to a common limit. What limit?

4. Let $[a_n, b_n]$, $n = 1, 2, \ldots$, be closed intervals with $[a_n, b_n] \cap [a_m, b_m] \neq \emptyset$ for all n, m.
Prove that $\bigcap_{n=1}^{\infty}[a_n, b_n] \neq \emptyset$.

5. The real sequence (a_n) is bounded but does not converge. Prove that it has two convergent subsequences with different limits.

6. Investigate the convergence of the following series. For those expressions containing the complex number z, find those z for which convergence occurs.

$$\sum_n \frac{\sin n}{n^2} \qquad \sum_n \frac{n^2 z^n}{5^n} \qquad \sum_n \frac{(-1)^n}{4+\sqrt{n}} \qquad \sum_n \frac{z^n(1-z)}{n}$$

7. Show that $\sum 1/(n \log^\alpha n)$ converges if $\alpha > 1$ and diverges otherwise.
Does $\sum 1/(n \log n \log\log n)$ converge?

8. Let $a_n \in \mathbb{C}$ and let $b_n = \frac{1}{n}\sum_{i=1}^{n} a_i$. Show that, if $a_n \to a$ as $n \to \infty$, then $b_n \to a$ also.

9. Consider the two series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ and $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots$, having the same terms but taken in a different order. Let s_n and t_n be the corresponding partial sums to n terms. Show that $s_{2n} = H_{2n} - H_n$ and $t_{3n} = H_{4n} - \frac{1}{2}H_{2n} - \frac{1}{2}H_n$, where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n}$. Show that s_n converges to a limit s and that t_n converges to $3s/2$.

10. Suppose that $\sum a_n$ diverges and $a_n > 0$. Show that there exist b_n with $b_n/a_n \to 0$ and $\sum b_n$ divergent.

11. Let $z \in \mathbb{C}$. Show that the series

$$\frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \frac{z^4}{1-z^8} + \frac{z^8}{1-z^{16}} + \cdots$$

converges to $z/(1-z)$ if $|z| < 1$, converges to $1/(1-z)$ if $|z| > 1$, and diverges if $|z| = 1$.

12. Prove that every real sequence has a monotonic subsequence. Deduce the Bolzano-Weierstrass theorem.

13. Can we write the open interval (0,1) as a disjoint union of closed intervals of positive length?

14. Is there an enumeration of $\mathbb{Q}$ as $q_1, q_2, q_3, \ldots$ such that $\sum(q_n - q_{n+1})^2$ converges?