Analysis I: Example Sheet 4

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1. Use the finite Taylor expansion to show that if f is a polynomial of degree n, then

$$f(x) = f(a) + (x-a)f^{(1)}(a) + \frac{(x-a)^2}{2!}f^{(2)}(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a).$$

2. Suppose that the functions f, g and h are continuous on $[a,b]$ and differentiable on (a,b). Show that there is some c in (a,b) such that

$$\begin{vmatrix} f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \\ f'(c) & g'(c) & h'(c) \end{vmatrix} = 0.$$

Expand the determinant to obtain an alternative formulation of this fact. What does this reduce to if $h(x) = 1$ for all x? What does this reduce to if $h(x) = 1$ and $g(x) = x$ for all x?

3. Suppose that $f : I \to \mathbb{R}$ is twice-differentiable, where $I = (0, +\infty)$, and let

$$M_0 = \text{lub}\{|f(x)| : x \in I\}, \quad M_1 = \text{lub}\{|f'(x)| : x \in I)\}, \quad M_2 = \text{lub}\{|f''(x)| : x \in I\}.$$

Show that if each M_j is positive and finite, then $M_1^2 \leqslant 4M_0M_2$.
Hint: Taylor's Theorem shows that (in a suitable situation)

$$f'(x) = \frac{f(x+2h) - f(x)}{2h} - hf''(c).$$

Now find an upper bound for $|f'(x)|$.

4. In each of the following cases, find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$:

$$a_n = n, \qquad a_n = \frac{n^3}{n!}, \qquad a_n = n^{\sqrt{n}}, \quad a_n = \left(\frac{n}{n+1}\right)^n, \qquad a_n = n!, \qquad a_n = [3 + (-1)^n]^n.$$

5. Show that if $|a_n|^{1/n} \to 1/R$ as $n \to \infty$ then $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R. Which of the examples in Question 4 can be solved by this method?

6. Suppose that the power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R.
(i) What can you say (about R) if you know that $|a_n| \leqslant M$ for all n?
(ii) What can you say if you know that $|a_n| \geqslant K > 0$ for infinitely many n?
(iii) What can you say if you know that both (i) and (ii) hold?

7. Let $F(x) = x - x^2/2 + x^3/3 - x^4/4 + \cdots$ with radius of convergence R. Show that $R = 1$. Show that if $|x| < 1$ then $(d/dx)\big[F(x) - \log(1+x)\big] = 0$. Deduce that if $-1 < x < 1$ then

$$\log(1+x) = x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

The left–hand side is defined for $x > -1$; the right–hand side is only defined for $|x| < 1$; this shows a weakness in the nature of power series.

8. Show that the power series $\sum_{n=1}^{\infty} z^n/n$ has radius of convergence 1. Show also that the series converges at every point of the unit circle $|z| = 1$ **except** at the point $z = 1$.
[Hint: take $|z| = 1$ and $z \neq 1$ and consider $(1-z)\sum_{n=1}^{m}(z^n/n)$. Note that $\sum_n 1/n(n+1)$ converges.]

9. Let $f(x) = x^2$. By taking a partition of $[0, a]$ with intervals of equal length show (directly from the definition of the integral) that $\int_0^a f = a^3/3$.

10. The rule for a change of variable in the integral is often stated (without comment) as

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(t))g'(t)dt.$$

Is this always true when $f(x) = 1/(1 + x^2)$ and $g(x) = 1/x$?

11. Show that

$$\int_2^3 \frac{dx}{(x-1)^2(x^2+1)} = \log\left[\left(\frac{e}{2}\right)^{1/4}\right].$$

12. Show that $\int_0^1 xe^x\, dx = 1$ (i) by integration by parts, and (ii) by using power series. *You should sum the series in* (ii), *and also justify each step that you take.*

13. Let $f(1) = 1$ and

$$f(x) = (n+2)\big[(n+1)x - n\big] \quad \text{when} \quad \frac{n}{n+1} \leqslant x < \frac{n+1}{n+2}, \qquad n = 0, 1, \ldots.$$

Show that f is integrable on $[0, 1]$, and evaluate $\int_0^1 f$. Can you 'see' this from the graph of f?

14. The Mean Value Theorem for Integrals Show that if f is continuous on $[a, b]$ then there is some c in $[a, b]$ such that

$$\frac{1}{b-a}\int_a^b f(x)\, dx = f(c).$$

Show that this need not be true if f is merely integrable on $[a, b]$.

15. Give an example of a function f for which $|f|^3$ is integrable on $[0, 1]$, but f is not.

16. Suppose that $f : [a, b] \to \mathbb{R}$ is integrable and that $f(x) > 0$ for every x in $[a, b]$. It should be clear that $\int_a^b f \geqslant 0$: explain why this is so. In fact, $\int_a^b f > 0$ but this is quite difficult to prove. Prove the simpler result that if f is continuous (and not merely integrable) then $\int_a^b f > 0$.

17. Suppose that $f : \mathbb{R} \to \mathbb{R}$ has a continuous derivative, $f(0) = 0$ and $|f'(x)| \leqslant M$ on $[0, 1]$. Show that $|\int_0^1 f| \leqslant M/2$. Show that if, in addition, $f(1) = 0$ then $|\int_0^1 f| \leqslant M/4$. What could you say if $|f'(x)| \leqslant M|x|$?

18. Which (if any) of the following functions integrable on $[0, 1]$?

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ x\sin(1/x) & \text{if } 0 < x \leqslant 1; \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is irrational}, \\ 1/q & \text{if } x = p/q \text{ with } p \text{ and } q \text{ coprime}; \end{cases}$$

$$h(x) = \begin{cases} x & \text{if } x \text{ is rational}, \\ x^2 & \text{if } x \text{ is irrational}. \end{cases}$$

19. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Let $f_0(x) = f(x)$, and, for $x \geqslant 0$ and $n = 1, 2, \ldots$,

$$f_{n+1}(x) = \frac{1}{n!}\int_0^x (x-t)^n f(t)dt.$$

Show that the n-th derivative of f_n exists and is equal to f.