Analysis I: Example Sheet 3

AFB, Lent 2006

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1. Let $f(x) = (x^2 + 2x + 1)/(x^2 + 2)$. Prove that f is bounded on $\mathbb{R}$. Find the least upper bound, and the greatest lower bound, of $\{f(x) : x \in \mathbb{R}\}$. Are either of these attained by f?

2. Let

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ -x & \text{if } x \text{ is irrational} \end{cases}.$$

At which points is f (i) continuous, and (ii) discontinuous?

3. We ask the following question: *suppose that a point w is not on a curve γ: can the curve approach arbitrarily close to w?* A *curve* γ in the complex plane is a continuous function from a *bounded real interval*, say $\{t : a \leqslant t \leqslant b\}$ into $\mathbb{C}$. Suppose that γ is a curve in $\mathbb{C}$, and that w is not on γ (this means that $\gamma(t) \neq w$ for any t in $[a, b]$). Show that there is a disc D with centre w, and of positive radius, such that $\gamma(t) \notin D$ when $a \leqslant t \leqslant b$. Thus *if w is not on γ, then γ cannot approach arbitrarily close to w.*

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous at every point of $\mathbb{R}$, and periodic (that is, for some positive p, and all x, $f(x + p) = f(x)$). Show that f has a maximum value, and a minimum value.

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x + y) = f(x) + f(y)$ for all real x and y. Show that $f(0) = 0$, and that if r is rational and x is real, then $f(rx) = rf(x)$. Show that f is continuous at 0 if and only if f is continuous at every point of $\mathbb{R}$. Show that if f is continuous at some point of $\mathbb{R}$, then $f(x) = ax$ for some constant a. [Be careful: there are solutions of this functional equation which are not of the form $f(x) = ax$. Such solutions are not continuous anywhere.]

6. Let f be a real-valued function defined on an interval I of the real line. The graph of f is said to have a *horizontal chord of length* a, where $a > 0$, if there is some x such that $x \in I$, $x + a \in I$ and $f(x + a) = f(x)$. Interpret this on a diagram. Show that the graph of $\sin x$ has, for every positive a, a horizontal chord of length a.

Prove the *Universal Chord Theorem*: if f is continuous on $[0, 1]$, and $f(0) = f(1)$, then the graph of f has horizontal chords of lengths $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$. However, if $0 < a < 1$ and a is not the reciprocal of an integer, then there is some continuous $f : [0, 1] \to \mathbb{R}$ with $f(0) = f(1)$, but which does not have a horizontal chord of length a. (A remarkable result!)
Hint. Suppose we want to show that f has a horizontal chord of length $1/3$. Let $g(x) = f(x + 1/3) - f(x)$; we want to show that g has a zero. Note that $[f(1/3) - f(0)] + [f(2/3) - f(1/3)] + [f(1) - f(2/3)] = 0$.
For the second part, suppose that $0 < a < 1$ and that a is not the reciprocal of an integer, and consider the function

$$h(x) = x - \frac{\sin^2(\pi x/a)}{\sin^2(\pi/a)}.$$

7. Consider the real interval $[0, 1]$ in the complex plane $\mathbb{C}$. For each z in $\mathbb{C}$, let

$$d(z) = \text{glb}\{|z - x| : x \in [0, 1]\};$$

thus $d(z)$ measures the distance from z to the point of $[0, 1]$ that is nearest to z. Prove that d is continuous at every point of $\mathbb{C}$, and that $d(z) = 0$ if and only if $z \in [0, 1]$. Would these two conclusions still hold if we used $(0, 1)$ instead of $[0, 1]$?

8. Limits of continuous functions are perhaps not as simple as you might think, so be careful. Here are two examples; *only attempt the first of these.*

(a) Show that for each real x,

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}(\cos m!\pi x)^n\right) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

(b) It can be shown that for each positive integer N,

$$\lim_{r\to\infty}\lim_{m\to\infty}\lim_{n\to\infty}\left(\sum_{k=0}^{m}\left[1-(\cos[(k!)^r\pi/N])^{2n}\right.\right)$$

exists and is the largest prime factor of N.

See G.H.Hardy, A formula for the prime factors of any number, *Messenger of Math.* 35 (1906), 145-146.

9. Suppose that f is a real-valued function defined on an interval (a,b), and suppose that $a<c<b$. Consider the two statements

(S1) $|f(c+x)-f(c)|\to 0$ as $x\to 0$;

(S2) $|f(c+x)-f(c-x)|\to 0$ as $x\to 0$.

Show that one of these statements implies the other, but that the statements are not equivalent to each other.

10. If $a>0$ and $b\in\mathbb{R}$ we define a^b to be $\exp(b\log a)$ (you may assume familiarity with the usual properties of exp and log). For non-zero real x, let $f(x)=|x|^b$, and let $f(0)=0$. Find the set of points x at which f is continuous. Find the set of points x at which f is differentiable. When f is differentiable at x, find $f'(x)$. [**NB**: these sets may depend on b.]

11. Derive Leibnitz' formula for the n-th derivative of a product: if $h(x)=f(x)g(x)$ then (assuming all derivatives exist)

$$h^{(n)}(x)=\sum_{k=0}^{n}\binom{n}{k}f^{(k)}(x)g^{(n-k)}(x).$$

[You may wish to use induction.] There is also an analogous formula for the n-th derivative of the composite function $f\big(g(x)\big)$, but this is more complicated. Derive the formula for, say, $n=1,2,3$. Can you suggest what the formulae might be?

12. Let $f(x)=x^3-x^2+2x-4$. According to the Mean Value Theorem there is at least one t in $[1,3]$ such that $f(3)-f(1)=(3-1)f'(t)$. Find one such value of t.

13. Show that if $0<a<b$ then $\frac{b-a}{1+b^2}<\tan^{-1}(b)-\tan^{-1}(a)<\frac{b-a}{1+a^2}$.

14. This question shows that a derivative $f'(x)$ need not be continuous. Define $f:\mathbb{R}\to\mathbb{R}$ by $f(0)=0$ and, for $x\neq 0$, $f(x)=x^2\sin(1/x)$. Show that $f'(x)$ exists for all real x, but that it is NOT a continuous function of x. [You may assume a complete knowledge of the function sin].

15. Suppose that $a<b<c$ and let $f(x)=(x-a)^3(x-b)^3(x-c)^3$. Show that the third derivative $f^{(3)}(x)$ has three distinct zeros in in (a,b) and three distinct zeros in (b,c).

[It is clear that $f^{(3)}(x)$ has at most six real zeros, for it is a polynomial of degree six. You have to show that all of these zeros are real and are located as given. Do NOT compute the third derivative; instead, use Rolle's Theorem to locate the eight zeros of f'.]

16. Show that the function $f(x)=x^2\sin(1/x)+\frac{1}{2}x$ for $x\neq 0$, and $f(0)=0$, has the following properties:

(a) $f'(0)$ exists and **is positive**;

(b) every open interval that contains 0 contains a smaller interval on which f is **strictly decreasing**.

Thus $f'(0)>0$ does NOT imply that f is increasing on some interval containing 0.