Analysis I: Example Sheet 2

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1. Given a sequence $a_1, a_2, \ldots$, let $s_n = a_1 + \cdots + a_n$. We say

- that the series $\sum_n a_n$ **converges** to S when $s_n \to S$ as $n \to \infty$, and
- that the sequence $a_1, a_2, \ldots$ is **summable** to S^* when $(s_1 + \cdots + s_n)/n \to S^*$ as $n \to \infty$.

(a) Show that if $\sum_n a_n$ converges to S, then $a_1, a_2, \ldots$ is summable to S.
(b) By taking $a_n = (-1)^n$ show that $a_1, a_2, \ldots$ may be summable even when $\sum_n a_n$ diverges.

It follows that the class of summable sequences includes (in an obvious sense) the class of convergent series (with the same answer), and moreover, it extends the class of sequences that we can handle. You may now ask why we don't always work with summable sequences instead of convergent series!

2. The theory of **infinite products** $b_1 b_2 \cdots$, or $\prod_{n=1}^{\infty} b_n$, is more subtle than the theory of infinite sums. Here is a start to the theory. It is natural to start with the definition that the infinite product $b_1 b_2 b_3 \cdots$ converges if the finite product $b_1 \cdots b_n$ converges as $n \to \infty$. This has the disadvantage that the infinite product will converge whenever **some** b_n is zero. Henceforth we shall suppose that $b_n \neq 0$ for all n. Now suppose that $b_1 \cdots b_n$ does converge, say to B. Then

$$b_{n+1} = \frac{b_1 \cdots b_n b_{n+1}}{b_1 \cdots b_n} \to \frac{B}{B} = 1$$

as $n \to \infty$. This is the analogue of $a_n \to 0$ for a convergent series $\sum_n a_n$.

For the rest of this question we suppose that $a_1, a_2, \ldots$ are non-negative numbers, and we let

$$s_n = a_1 + \cdots + a_n, \quad p_n = (1 + a_1) \cdots (1 + a_n).$$

Use the inequality $1 + x \leqslant e^x$ for positive x (and standard properties of the exponential function – which will be derived later in this course) to show that $s_n \leqslant p_n \leqslant e^{s_n}$. Use this to prove the following

Theorem. *Suppose that $a_n \geqslant 0$ for all n. Show that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\prod_{n=1}^{\infty}(1 + a_n)$ converges.*

According to this result the infinite products

$$\prod_{n=2}^{\infty} \left(1 + \frac{1}{n^2 - 1}\right), \quad \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)$$

converge and diverge, respectively. Give direct proofs of these facts, and find the value of the first of these infinite products.

Questions on upper and lower bounds

3. Suppose that a subset E of $\mathbb{R}$ has a **maximal element** e (that is, $x \leqslant e$ for every x in E). Prove (formally) that $\operatorname{lub} E = e$.

4. Let $P(z) = (z - a_1)(z - a_2)(z - a_3)(z - a_4)$ and $Q(z) = (z - a_1)(z - a_2)(z - a_3)(z - a_4)(z - a_5)$, where $a_1 < a_2 < a_3 < a_4 < a_5$. Determine which of the sets

$$\{x \in \mathbb{R} : P(x) < 0\}, \quad \{x \in \mathbb{R} : P(x) > 0\}, \quad \{x \in \mathbb{R} : Q(x) < 0\}, \quad \{x \in \mathbb{R} : Q(x) > 0\}$$

are (i) bounded above, (ii) bounded below. When one of these bounds exists, find the least upper bound or greatest lower bound as appropriate.

5. Is the following assertion true or false?
A non-empty subset E of $\mathbb{R}$ is bounded above if and only if every non-empty subset of E is bounded above.

6. Let P be the parabola given by the equation $y = x^2$ (so that $x + iy \in P$ if and only if $y = x^2$), and let $z_0 = 3 + 7i$. Find $\text{glb}\{|z - z_0| : z \in P\}$.

7. Suppose that A and B are non-empty subsets of $\mathbb{R}$. Show that if, for all a in A, and all b in B, $a < b$ then $\text{lub}A \leqslant \text{glb}B$. Give an example in which $\text{lub}A = \text{glb}B$.

8. Suppose that A and B are non-empty sets of real numbers, each bounded above, and define

$$A + B = \{a + b : a \in A,\ b \in B\}, \quad AB = \{ab : a \in A,\ b \in B\}.$$

Show that $A + B$ is non-empty and bounded above. Is it true that $\text{lub}\,(A + B) = \text{lub}\,(A) + \text{lub}\,(B)$? Show that AB need not be bounded above. Is it true that if AB is bounded above then $\text{lub}\,AB = \text{lub}\,A \times \text{lub}\,B$?

9. Let a_n be a real sequence. Show that $a_n \to a$ if and only if for every pair of real numbers α and β with $\alpha < a < \beta$, there is an n_0 such that $n > n_0$ implies that $\alpha < a_n < \beta$.
[*This definition of convergence uses only the ordering of $\mathbb{R}$, and not the distance on $\mathbb{R}$. Because of this, it generalizes easily to give the appropriate definitions of $x_n \to +\infty$ and $x_n \to -\infty$. For example, $x_n \to +\infty$ if and only if for every real α there is an n_0 such that $n > n_0$ implies $x_n > \alpha$.*]

Questions on continuous functions

10. Let E be a non-empty subset of $\mathbb{C}$. Suppose that $a_1, \dots a_n$ are complex numbers, and that $f_1, \dots, f_n$ are complex-valued functions that are defined and continuous at every point of E. Show that $a_1 f_1 + \cdots + a_n f_n$ is continuous at every point of E.
[*Question 2 on Sheet 1 gives a set E, and functions $f_1, f_2, \dots$, each continuous on E, such that the* **infinite** *convergent series $\sum_{n=1}^{\infty} f_n(z)$ is* **not** *continuous on E*].
Deduce that if $f(z) = \sum_{m=0}^{p} \sum_{n=0}^{q} a_{m,n} x^m y^n$, where $z = x + iy$ (with x and y real), and the $a_{i,j}$ are real numbers, then f is continuous on $\mathbb{C}$.

11. In each of the following cases decide whether the function f, which is defined on $\mathbb{R}$ and has $f(0) = 0$, is continuous at 0. Justify your answers.
(a) $f(x) = x \sin(1/x)$ when $x \neq 0$;
(b) $f(x) = \sin(1/x)$ when $x \neq 0$;
(c) $f(x) = (1/x)\sin(1/x)$ when $x \neq 0$;
(d) $f(x) = x$ if x is rational, and $f(x) = -x$ if x is irrational.

12. The *ruler function* f (compare the graph of f with the markings on a ruler in inches) is defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is an integer,} \\ 1/2^k & \text{if } x = p/2^k \text{ for some odd integer } p \text{ and some non-negative integer } k, \\ 0 & \text{otherwise.} \end{cases}$$

At which points is f (i) continuous, and (ii) discontinuous?

13. Suppose that $f : [a, b] \to \mathbb{R}$ is strictly increasing, and let $E = [a, b]$ and $f(E) = \{f(x) : x \in E\}$. Show that
(a) $f^{-1} : f(E) \to E$ is continuous on $f(E)$ **regardless of whether** $f : E \to f(E)$ **is continuous or not**;
(b) $f : E \to f(E)$ is continuous on E if and only if $f(E)$ is an interval.
