Analysis I: Example Sheet 1

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1. Discuss the convergence of the series $\sum_{n=0}^{\infty}\left(z^n - z^{n+1}\right)$, where z is complex. What is its sum when it converges?

2. Find all real values x for which the series

$$x + \frac{x}{1+x} + \frac{x}{(1+x)^2} + \frac{x}{(1+x)^3} + \cdots$$

converges and, when it converges, denote its value by $S(x)$. Sketch the graph of $S(x)$. Use this example to show (without worrying about a formal definition of continuity at this stage) that *an infinite sum of continuous functions need not be continuous.*

3. Consider the series

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}, \quad \sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}, \quad \sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)(n+3)}.$$

By considering the expression $1/n - 1/(n+1)$ show that the first series converges to 1. By considering other partial fractions, find the value of the other two series.

Show that for $k = 1, 2, \ldots,$

$$\frac{1}{n(n+1)\cdots(n+k)} = \frac{1}{k!}\sum_{r=0}^{k}\binom{k}{r}\frac{(-1)^r}{n+r}.$$

4. Let $a_1, a_2, \ldots$ be a sequence of real numbers, and define the sequence $b_1, b_2, \ldots$ by $b_1 = a_1 + a_2$, $b_2 = a_3 + a_4$, $b_3 = a_5 + a_6$, and so on.
(i) Show that if $a_1 + a_2 + a_3 + \cdots$ converges then so does $b_1 + b_2 + b_3 + \cdots$, and the two sums are the same. [*This shows that we can always insert brackets into a convergent series without changing its sum*].
(ii) Give an example in which $b_1 + b_2 + b_3 + \cdots$ converges yet $a_1 + a_2 + a_3 + \cdots$ diverges. [*This shows that we cannot always remove brackets from a convergent series without changing its sum.*]

5. Determine whether each of the following series converges or diverges. In each case, if the series converges (i) obtain a simple upper bound for its sum, and (ii) use a computer to find an approximate value for its sum.

$$\sum_{n=1}^{\infty}\frac{2n}{5n^3 - n + 6}, \qquad \sum_{n=2}^{\infty}\frac{n^{n/2}-1}{n^n - 1}, \qquad \sum_{n=1}^{\infty}\frac{n^2}{3^n}, \qquad \sum_{n=1}^{\infty}(|\sin n| + |\cos n|).$$

6. Suppose that $x_1, x_2, \ldots$ are positive, and let

$$S_1 = \sum_{n=1}^{\infty}\sqrt{x_n}, \quad S_2 = \sum_{n=1}^{\infty}\sqrt{\frac{x_n}{1+x_n}}, \quad S_3 = \sum_{n=1}^{\infty}\sqrt{x_n x_{n+1}}.$$

(i) Show that if S_1 converges then S_2 and S_3 also converge.
(ii) If S_2 converges does S_1 necessarily converge?
(iii) If S_3 converges does S_1 necessarily converge?

7. For $n = 1, 2, \ldots$, let

$$a_n = \frac{1}{\sqrt{n}} + \frac{(-1)^{n-1}}{n}.$$

Show (i) each a_n is positive, (ii) $a_n \to 0$ as $n \to \infty$, and (iii) $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ diverges. [This shows that in the Alternating Series Test for $\sum_n (-1)^n a_n$ it is essential to check that $|a_1| \geqslant |a_2| \geqslant \cdots$.]

8. Let

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}, \qquad b_n = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}.$$

Show that one of these sequences is increasing while the other is decreasing. Deduce that both sequences converge to some real number A. Use a computer to estimate A.

It will follow from our work later in the course that

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n$$

converges to some value γ (which is called *Euler's constant*). Use this fact to evaluate A.
[*Note that a_n does not converge to 0 even though each term in the expression for a_n does converge to 0.*]

9. Suppose that $k > 1$ and also that the real numbers $a_0, a_1, \ldots$ satisfy

$$a_{n+2} - (k + k^{-1})a_{n+1} + a_n = 0, \quad n \geqslant 0.$$

Given that $a_0 = 1$ find all values of a_1 such that the sequence a_n converges.

10. Sketch the graph of $f(x) = x^2 - 2x + 2$.

Suppose that we are given x_1, and that we define the sequence x_n inductively by $x_{n+1} = f(x_n)$ or, equivalently, by

$$x_{n+1} = x_n^2 - 2x_n + 2, \quad n = 1, 2, \ldots.$$

Make a conjecture about the behaviour of x_n as $n \to \infty$ in each of the cases (i) $1 < x_1 < 2$, (ii) $x_1 = 2$, (iii) $x_1 > 2$. Now prove your conjectures.
[You might also predict the behaviour in each case by evaluating $x_1, \ldots, x_{100}$, say, on a computer for selected values of x_1.]

11. This question is important (and not difficult).
Let

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots, \quad S^* = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots.$$

The construction of S^* is as follows. We add the first two terms of $1, 1/3, 1/5, 1/7, 1/9, \ldots$ and then subtract the first term of $1/2, 1/4, 1/6, \ldots$. Next, we add the next two terms of $1, 1/3, 1/5, 1/7, 1/9, \ldots$, and then subtract the next term of $1/2, 1/4, 1/6, \ldots$, and so on in the obvious way. Notice that *S and S^* have the same terms, but they are added together in a different order.* Notice also that S converges, *but we do not yet know whether S^* converges or diverges.*

Let S_n and S_n^* be the sum of the first n terms of S and S^*, respectively, and let

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Prove (for example, by induction) that

$$S_{2n} = H_{2n} - H_n, \quad S_{3n}^* = H_{4n} - \tfrac{1}{2}H_{2n} - \tfrac{1}{2}H_n.$$

Deduce that $S_{3n}^* = S_{4n} + \frac{1}{2}S_{2n}$, and hence show (i) that S^* converges, and (ii) that $2S^* = 3S$. Now show that $S \neq 0$ and hence that $S \neq S^*$. Thus *although S and S^* have the same terms, they sum to different values*! Note that according to Question 8, $S_{2n} = a_n$ so that $S = \log 2$ and $S^* = (3/2)\log 2$.