Analysis I: Example Sheet 1

AFB, Michaelmas 2004

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Question 1 Suppose that $x \geqslant 0$. Show that

$$x + \frac{x}{1+x} + \frac{x}{(1+x)^2} + \frac{x}{(1+x)^3} + \cdots = \begin{cases} 0 & \text{if } x = 0, \\ 1+x & \text{if } x > 0. \end{cases}$$

[Without (at this stage) worrying about a formal definition of continuity, this shows that an infinite sum of continuous functions need not be continuous.]

Question 2 By considering the expression $1/n - 1/(n+1)$ show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$. Find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$. Does this idea generalize ?

Question 3 Consider the two series $S = a_1 + a_2 + a_3 + \cdots$ and $S' = (a_1 + a_2) + (a_3 + a_4) + \cdots$. Formally, $S' = b_1 + b_2 + \cdots$, where $b_n = a_{2n-1} + a_{2n}$. Show that if S converges then so does S'. Give an example of a_i in which S' converges yet S diverges.
[This shows that we can always insert brackets into a convergent series *but we cannot always remove them* !]

Question 4 Suppose that $x_1, x_2, \ldots$ are positive, and let

$$S_1 = \sum_{n=1}^{\infty} \sqrt{x_n}, \qquad S_2 = \sum_{n=1}^{\infty} \sqrt{\frac{x_n}{1+x_n}}.$$

Show that S_1 converges if and only if S_2 converges. Let

$$S_3 = \sum_{n=1}^{\infty} x_n, \qquad S_4 = \sum_{n=1}^{\infty} \sqrt{x_n x_{n+1}}.$$

If S_3 diverges must S_4 diverge? If S_4 diverges must S_3 diverge?

Question 5 Determine whether each of the following series converges or diverges. If a series converges try to obtain an upper bound for its sum.

$$\sum_{n=1}^{\infty} \frac{2n}{5n^3 - n + 6}, \qquad \sum_{n=1}^{\infty} \frac{1 + n^{n/2}}{1 + n^n}, \qquad \sum_{n=1}^{\infty} \frac{n^{2000000} + 3^{n/2}}{n + 3^n}, \qquad \sum_{n=1}^{\infty} |\sin n| + |\cos n|.$$

[Hint: use the Binomial Theorem to show that for positive integers n and k, $(3/2)^{n+k} > (n/2)^k (1/k!)$]

Question 6 Given x_1, the sequence x_n is defined inductively by the recurrence relation

$$x_{n+1} = x_n^2 - 2x_n + 2, \quad n = 1, 2, \ldots.$$

Make a conjecture about the behaviour of x_n as $n \to \infty$ in each of the cases (i) $1 < x_1 < 2$, (ii) $x_1 = 2$, (iii) $x_1 > 2$. Now prove your conjectures.
[Let $f(x) = x^2 - 2x + 2$; then $x_{n+1} = f(x_n)$, so it is clear that it might help to sketch the graph of $y = f(x)$. It might also help to evaluate $x_1, \ldots, x_{100}$, say, on a computer for selected values of x_1.]

Question 7 Suppose that $p > 0$ and $q > 0$. Given a_0, define a_n and b_n inductively by

$$a_0 = p, \qquad a_{n+1} = \frac{1}{2}\left(a_n + \frac{q^2}{a_n}\right), \qquad b_n = \frac{q^2}{a_n}.$$

Show that the sequences $a_1, a_2, \dots$ and $b_1, b_2, \dots$ are monotonic and bounded. Show also that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, and find this common value.
[Should you worry about whether $a_n = 0$ for some n? Put $f(x) = \frac{1}{2}(x + q^2/x)$ and sketch the graph of f. Interpet the identity $f(q^2/x) = f(x)$ in the context of this question.]

Question 8 (this question is important, and well worth trying) Let

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots, \quad S' = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots.$$

It is important to understand how S' is constructed. We add the first two terms of $1, 1/3, 1/5, 1/7, 1/9, \dots$, then subtract the first term of $1/2, 1/4, 1/6, \dots$, then add the next two terms of $1, 1/3, 1/5, 1/7, 1/9, \dots$, then subtract the next term of $1/2, 1/4, 1/6, \dots$, and so on in the obvious way. Thus S and S' have *the same terms*, but *they are added in a different order*. We are going to see that $S = \log 2$ and $S' = (3/2)\log 2$; thus $S \neq S'$ so that *the order of the terms in an infinite sum does matter* ! We let

$$S_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Verify the following steps (you may assume knowledge of the logarithm).

Step 1 As $1/(n+1) \leqslant 1/x \leqslant 1/n$ for $n \leqslant x \leqslant n+1$ we have

$$\frac{1}{n+1} \leqslant \log\left(\frac{n+1}{n}\right) \leqslant \frac{1}{n}.$$

Step 2 Deduce that $\log(n+1) \leqslant S_n \leqslant 1 + \log n$.

Step 3 Put $T_n = S_n - \log n$. Deduce that $T_1 \geqslant T_2 \geqslant T_3 \geqslant \cdots \geqslant T_n \geqslant \cdots \geqslant 0$, and hence that T_n converges, say to γ (which is called *Euler's constant*). We write this as

$$S_n = \log n + \gamma + \varepsilon_n, \quad \varepsilon_n = T_n - \gamma \to 0.$$

Step 4 Show that

$$1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{2n} = S_{2n} - S_n,$$

and use Step 3 to show that the series S converges to the value $\log 2$.

Step 5 Show that the sum of the first $3n$ terms of S' is $S_{4n} - \frac{1}{2}S_{2n} - \frac{1}{2}S_n$, and use Step 3 to show that S' converges to the value $(3/2)\log 2$. Thus $S \neq S'$ - BE WARNED!

Question 9 For each n, let S_n be the square in the plane $\mathbb{R}^2$ given by $\{(x,y) : a_n \leqslant x \leqslant b_n,\ c_n \leqslant y \leqslant d_n\}$. Show that if $S_1 \supset S_2 \supset S_3 \supset \cdots$ then $\cap_{n=1}^{\infty} S_n$ is a non-empty square of the form

$$\{(x,y) : a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}.$$

Now suppose that T_n is *any* non-empty square in the plane (including its boundary, and not necessarily with its sides parallel to the axes), and that $T_1 \supset T_2 \supset T_3 \supset \cdots$. Use the Bolzano-Weierstrass Theorem to show that $\cap_{n=1}^{\infty} T_n$ is non-empty.

Question 10 Let θ be a real number. Discuss the behaviour of the sequence $\cos(n\theta)$ as $n \to \infty$. You are not required to give proofs, but you are required to collect evidence, speculate, and so on. You may wish to compare this to the sequence $e^{in\theta}$ of complex numbers that lie on the unit circle.

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Question 1 Suppose that A and B are non-empty sets of real numbers, each bounded above, and define

$$A+B=\{a+b : a \in A,\ b \in B\}, \quad AB=\{ab : a \in A,\ b \in B\}.$$

Show that $A+B$ is non-empty and bounded above, and that $\mathrm{lub}(A+B)=\mathrm{lub}(A)+\mathrm{lub}(B)$. Discuss the corresponding assertion for the set AB.

Question 2 Suppose that $a<b<c<d$, and let $E=\{x \in \mathbb{R} : (x-a)(x-b)(x-c)(x-d)<0\}$. Show that E is non-empty and bounded above, and find $\mathrm{lub}(E)$ and $\mathrm{glb}(E)$. Do either of these belong to E?

Question 3 Suppose that z is a complex number and R is positive. Let $E=\{|z-w| : |w|=R\}$. Give $\mathrm{lub}(E)$ and $\mathrm{glb}(E)$ and prove your result.

Question 4 Finding the greatest lower bound of a set is not always easy. Let P be the parabola given by the equation $y=x^2$ (so that $x+iy \in P$ if and only if $y=x^2$), let $z=3+7i$, and let $E=\{|z-w| : w \in P\}$. A quick calculation gave $\mathrm{glb}(E)=0.34811$. Do you agree?

Question 5 (i) For which positive values x does the series $\sum_{n=1}^{\infty} x^n n^x$ (a) converge? (b) diverge? [You may assume reasonable properties of the function n^x.]

(ii) Does $\sum_{n=1}^{\infty}\left(\sqrt{n^2+1}-n\right)$ converge or diverge?

(iii) Is

$$\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{\sqrt{n}}+i\frac{1}{n^2}\right)$$

convergent or divergent? Is it absolutely convergent?

Question 6 We have defined the value of the infinite sum $\sum_{n=1}^{\infty}$ to be the $\lim_{n\to\infty} s_n$ (when it exists), where $s_n=a_1+\cdots+a_n$. However, this is not the only possible definition: another one is

$$\lim_{n\to\infty}\frac{s_1+\cdots+s_n}{n}$$

when this exists. This is called Cesàro summability.

(i) Show that if $\sum_n a_n$ converges to the value A, then $(s_1+\cdots+s_n)/n$ also tends to A (that is, the sequence a_n is Cesàro summable to the same value A).
[Hint: if $s_n \to A$, then for $n>n_0$, say, s_n is approximately A, so that $(s_1+\cdots+s_n)/n$ is approximately

$$\frac{s_1+\cdots+s_{n_0}+(n-n_0)A}{n},$$

and this is $A+\varepsilon_n$, where $\epsilon_n \to 0$. Now convert this into a rigorous argument.]

(ii) Let $a_n=(-1)^n$. Does $\sum_n a_n$ converge? Is it Cesàro summable?

(iii) Discuss the relative merits of convergence and Cesàro summability.

Question 7 We have discussed infinite sums $a_1+a_2+\cdots$. The theory of **infinite products** $b_1b_2\cdots$ (or $\prod_{n=1}^{\infty} b_n$) is much more subtle. However, here is a start to the theory. We suppose throughout that the a_n are positive numbers.

Let $s_n = a_1 + \cdots + a_n$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence s_n is bounded above.

Let $p_n = (1+a_1)\cdots(1+a_n)$; this is an increasing sequence, and we say that the infinite product $\prod_{n=1}^{\infty}(1+a_n)$ converges if and only if the sequence p_n converges. Show that if the infinite product $\prod_{n=1}^{\infty}(1+a_n)$ converges then the infinite sum $\sum_{n=1}^{\infty} a_n$ converges.

Show that for positive x, $1+x \leqslant e^x$. Now show that if the infnite sum $\sum_n a_n$ converges then the infinite product $\prod_n (1+a_n)$ converges. You have now proved the following

Theorem. *Suppose that* $a_n > 0$ *for all* n. *Then* $\sum_{n=1}^{\infty} a_n$ *converges if and only if* $\prod_{n=1}^{\infty}(1+a_n)$ *converges.*

According to this result the products

$$\prod_{n=2}^{\infty}\left(1+\frac{1}{n^2-1}\right), \quad \prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)$$

converge and diverge, respectively. Give direct proofs of these facts, and find the value of the first of these products.

Question 8 A **sequence** is a function from $\mathbb{N}$ to (say) $\mathbb{R}$. A **double sequence** is a function from $\mathbb{N}\times\mathbb{N}$ to (say) $\mathbb{R}$, where $\mathbb{N}\times\mathbb{N} = \{(m,n) : m,n \in \mathbb{N}\}$. For example, $a_{p,q}$ is a double sequence, where

$$a_{p,q} = \frac{p}{p+q}, \qquad p,q = 1,2,3,\ldots.$$

Is

$$\lim_{p\to\infty}\left(\lim_{q\to\infty} a_{p,q}\right) = \lim_{q\to\infty}\left(\lim_{p\to\infty} a_{p,q}\right)?$$

Let k be a positive integer. What is $\lim_{p\to\infty} a_{p,kp}$?

Question 9 Let x be a real number. Show that

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}(\cos m!\pi x)^{2n}\right) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1 & \text{if } x \text{ is rational.}\end{cases}$$

[For each m, put $y_m = \lim_{n\to\infty}(\cos m!\pi x)^{2n}$. Although this is elementary, it needs a little thought!]

Question 10 (a) Construct a sequence of rational numbers with the property that there is a subsequence converging to x if and only if $0 \leqslant x \leqslant 1$.
[*Hint* : consider k/n for $0 \leqslant k \leqslant n$ first for $n=1$, then $n = 2, 3\ldots$]
(b) Construct a sequence of rational numbers with the property that, for every real number x, there is a subsequence converging to x.
(c) Is it possible to construct a sequence of rational numbers with the property that there is a subsequence converging to x if and only if $x > 0$?

Question 11 Let a_n be a real sequence. Show that a_n converges to a if and only if for every pair of real numbers α and β with $\alpha < a < \beta$, there is an n_0 such that $n > n_0$ implies that $\alpha < a_n < \beta$.
This definition of convergence does not need the concept of distance, only order, and it generalizes easily to give the appropriate definitions of $x_n \to +\infty$ *and* $x_n \to -\infty$.

Analysis I: Example Sheet 3

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Question 1 Let E be a subset of $\mathbb{C}$. Suppose that $a_1, \ldots a_n$ are complex numbers, and that $f_1, \ldots, f_n$ are complex-valued functions that are defined on E, and that are continuous at every point of E. Show that $a_1 f_1 + \cdots + a_n f_n$ is continuous at every point of E.

Now review Exercise Sheet 1 where you will find an example of a set E, and functions $f_1, f_2, \ldots$, with the properties
(a) each f_n is continuous at every point of E,
(b) for each z in E the series $\sum_{n=1}^{\infty} f_n(z)$ converges, say to $f(z)$,
(c) f is not continuous at every point of E.

Question 2 Let $f(z) = \sum_{m=0}^{p} \sum_{n=0}^{q} a_{m,n} x^m y^n$, where $z = x + iy$ (with x and y real), and the $a_{i,j}$ are real constants (thus f is the general real polynomial in the two real variables x and y). Prove carefully that f is continuous on $\mathbb{C}$.

Question 3 In each of the following cases decide whether the function f, which is defined on $\mathbb{R}$, and has $f(0) = 0$, is continuous at 0. Justify your answer.
(a) $f(x) = x \sin(1/x)$ when $x \neq 0$;
(b) $f(x) = \sin(1/x)$ when $x \neq 0$;
(c) $f(x) = (1/x)\sin(1/x)$ when $x \neq 0$;
(d) $f(x) = x$ if x is rational, and $f(x) = -x$ if x is irrational.

Question 4 Prove that, given any real number a, there is some θ in the interval $[a, a + 2\pi]$ such that $\sin\theta = \left(\sqrt{1 + \sqrt{1 + \sqrt{2}}}\right)^{-1}$. [You may assume that the function sin is continuous on $\mathbb{R}$.]

Question 5 Let $E = [0, 1) \cup [2, 3] = \{x : 0 \leqslant x < 1\} \cup \{x : 2 \leqslant x \leqslant 3\}$, and let f be defined on E by

$$f(x) = \begin{cases} x & \text{if } x \in [0,1), \\ x - 1 & \text{if } x \in [2,3]. \end{cases}$$

Let $E' = \{f(x) : x \in E\}$. Show that $E' = [0, 2]$. Show that f is strictly increasing on E (so that f^{-1} exists on E'). Is f continuous on E? Is f^{-1} continuous on E'?

Question 6 Suppose that $f : [a, b] \to \mathbb{R}$ is strictly increasing (that is, if $a \leqslant x < y \leqslant b$ then $f(x) < f(y)$), and let $E = [a, b]$ and $E' = f\big([a, b]\big) = \{f(x) : a \leqslant x \leqslant b\}$.
(a) Show that $f^{-1} : E' \to E$ is continuous on E' *regardless of whether* $f : E \to E'$ *is continuous or not.*
(b) Show that $f : E \to E'$ is continuous on E if and only if E' is an interval.

Question 7 Let $f : \mathbb{R} \to [0, 1]$ be defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is an integer,} \\ 1/2^n & \text{if } x = p/2^n, \text{ where } p \text{ is an odd integer, and } n = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

This is sometimes called the **ruler function** (compare the the graph of f with the markings on a ruler in inches). At which points is f (i) continuous (ii) discontinuous?

Question 8 For each real x and positive α let $f(x) = |x|^{\alpha}$. For which real x, and positive α does the derivative $f'(x)$ exist?

Question 9 Supose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leqslant (x-y)^2$. Prove that f is constant on $\mathbb{R}$. What can you say if $(x-y)^2$ is replaced by $|x-y|^\alpha$, where $\alpha > 1$? What can you say if $(x-y)^2$ is replaced by $|x-y|$?

Question 10 Suppose that f is differentiable in the interval $(-1,1)$. Is it true that if $0 < a_n < b_n$, $a_n \to 0$ and $b_n \to 0$, then $\big(f(b_n) - f(a_n)\big)/(b_n - a_n) \to f'(0)$ as $n \to \infty$? [You should give a proof, or an explicit example to show that this is not true.]

Question 11 Define $f : \mathbb{R} \to \mathbb{R}$ by $f(0) = 0$ and, for $x \neq 0$, $f(x) = x^2 \sin(1/x)$ (You may assume a complete knowledge of the function sin). Show
(a) $f'(x)$ exists for all real x;
(b) $f'(x)$ is NOT a continuous function of x.

Question 12 Derive Leibnitz' formula for the n-th derivative of a product, namely that if $h(x) = f(x)g(x)$ then (assuming all derivatives here exist)

$$h^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x).$$

Question 13 Revise your notes (or consult a text) on L'Hospital's Rule. Now prove the following, and then comment on this example.

For $0 < x < 1$ let $f(x) = x$ and $g(x) = x + x^2 e^{i/x^2}$. Here, i is the usual complex number with $i^2 = -1$, and you may use the familiar properties of e^z. Show that
(a) $f(x) \to 0$ and $g(x) \to 0$ as $x \to 0$;
(b) $f'(x)/g'(x) \to 0$ as $x \to 0$;
(c) $f(x)/g(x) \to 1$ as $x \to 0$.

Question 14 Suppose that f is a real-valued function that is twice differentiable on the interval $(0, +\infty)$ [that is, $f''(x)$ exists for all positive x], and also that

$$M_0 = \text{lub}\{|f(x)| : x > 0\} \quad M_1 = \text{lub}\{|f'(x)| : x > 0\}, \quad M_2 = \text{lub}\{|f''(x)| : x > 0\}$$

(you may assume that these exists and are finite). Show that $(M_1)^2 \leqslant 2M_0M_2$.
[Hint: use Taylor's Theorem to obtain an expression for f' in terms of f and f''.]

Use this result to show that if f is twice differentiable on $(0, +\infty)$, and if $f(x) \to 0$ as $x \to +\infty$ then either $f'(x) \to 0$ or the rate of change of $f'(x)$ is unbounded.

Question 15 The *Schwarzian derivative* of f is defined to be

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

Show that $S_f(z) = 0$ for every f of the form $f(z) = (az+b)/(cz+d)$ (thus the Schwarzian derivative annihilates all Möbius maps, just as the ordinary derivative annihilates all constants). Can you find a 'simpler' combination of derivatives that annhilates all Möbius maps? (You are advised not to spend too long on this part of the question.)

Analysis I: Example Sheet 4 AFB, Michaelmas 2004

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Question 1 Let $f(x) = x^3 - x^2 + 2x - 4$. Find a value t in $[1, 3]$ such that $f(3) - f(1) = (3-1)f'(t)$ (i.e. verify the Mean Value Theorem in a specific case).

Question 2 Show that if $0 < a < b$ then

$$\frac{b-a}{1+b^2} \leqslant \tan^{-1}(b) - \tan^{-1}(a) < \frac{b-a}{1+a^2}.$$

Question 3 Evaluate the limits

$$\lim_{x\to 0} \big(\cos x\big)^{1/x^2}, \qquad \lim_{x\to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right).$$

Question 4 Let $f(x) = 1 - (1-x)^{2/3}$ where $0 \leqslant x \leqslant 2$. Clearly $f(0) = f(2) = 1$. Which of the two following (contradictory) statements is true and which is false?
(a) By Rolle's Theorem, there is some t in $[0, 2]$ with $f'(t) = 0$.
(b) $f'(x) \neq 0$ for any x in $[0, 2]$.

Question 5 Suppose that $a < b < c$ and let $f(x) = (x-a)^3(x-b)^3(x-c)^3$. Show that the third derivative $f^{(3)}(x)$ has three distinct zeros in in (a, b) and three distinct zeros in (b, c).
[It is clear that $f^{(3)}(x)$ has at most six real zeros, for it is a polynomial of degree six. You have to show that all zeros are real and are located as given. The solution uses only Rolle's Theorem!]

Question 6 Give an example in which $|f|$ is integrable on $[0, 1]$ but that f is not.

Question 7 Show (directly from the definition of the integral) that $\int_0^a t^2 dt = a^3/3$.
Evaluate

$$\int_{-1}^{1} t(t + |t|)\, dt.$$

Question 8 For $n = 0, 1, \ldots$, let $f(x) = (n+2)[(n+1)t - n]$ if $n/(n+1) \leqslant t \leqslant (n+1)/(n+2)$, and let $f(1) = 1$. Sketch the graph of f, and show that f is integrable on $[0, 1]$. Guess what $\int_0^1 f$ is, and then prove that your guess is correct (or not, as the case may be).

Question 9 Show that if f_1 and f_2 are integrable on $a, b]$ then so is $\max(f_1, f_2)$. [Hint: $\max(f_1, f_2)$ is the same as $f_2 + \max(f_1 - f_2, 0)$.] By contrast, show that if $f_1, f_2, \ldots$ are integrable on $[a, b]$, and if $f(x) = \sup_n f_n(x)$, then f need not be integrable on $[a, b]$.

Question 10 Suppose that f is differentiable on some open interval containing $[0, 1]$ with $|f'(x)| \leqslant M$ there, and that $f(0) = 0$. Show that $|\int_0^1 f| \leqslant M/2$. Show that if, in addition, $f(1) = 0$ then $|\int_0^1 f| \leqslant M/4$.

Question 11 The rule for a change of variable in the integral is usually stated as

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(t))g'(t)dt. \qquad (*)$$

Evaluate both sides of this when $f(x) = 1/(1+x^2)$, $g(x) = 1/x$, $a = -1$ and $b = 1$, and comment on your result.

Question 12 Evaluate the integral

$$\int_2^3 \frac{dx}{(x-1)^2(x^2+1)}.$$

Question 13 Show that

$$\int_0^1 xe^x\,dx = 1.$$

You could (I imagine) have done this at school. Here you should *justify each step that you take.*

Question 14 **The Mean Value Theorem for Integrals** Suppose that f is integrable on $[a,b]$. Is it true that for some c in $[a,b]$ we have

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)?$$

Show that this is true for **continuous** f, but that it need not be true for integrable f.

Question 15 Which (if any) of the following functions integrable on $[0,1]$?

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ x\sin(1/x) & \text{if } 0 < x \leqslant 1; \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1/q & \text{if } x = p/q \text{ with } p \text{ and } q \text{ coprime;} \end{cases}$$

$$h(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ x^2 & \text{if } x \text{ is irrational.} \end{cases}$$

In each case in which the function is integrable either evaluate or estimate the value of the integral.

Question 16 Let

$$I_n = \int \frac{x^n\,dx}{\sqrt{ax^2+2bx+c}}$$

(an indefinite integral). Show (stating any assumptions that you need to make) that

$$(n+1)aI_{n+1} + (2n+1)bI_n + ncI_{n-1} = x^n\sqrt{ax^2+2bx+c},$$

and hence find

$$\int \frac{x^2\,dx}{\sqrt{x^2+4x+3}}, \qquad \int \frac{dx}{x^2\sqrt{x^2+4}}.$$

Question 17 Suppose that f is continuous on $[0,a]$, put $f_0(x) = f(x)$, and

$$f_{n+1}(x) = \frac{1}{n!}\int_0^x (x-t)^n f(t)dt.$$

Show that the n-th derivative of f_n exists and is equal to f.

Question 18 Suppose that $f : [a,b] \to \mathbb{R}$ is integrable and that $f(x) > 0$ for every x in $[a,b]$. It should be clear that $\int_a^b f \geqslant 0$, and you should explain why this is so.

In fact, the stronger inequality $\int_a^b f > 0$ holds. This is easy if f is continuous on $[a,b]$; give a proof of this. The proof when f is not necessarily continuous is harder, and can be proved as follows.

(i) Suppose that $\lambda > 0$. Show that there is some interval $[c,d]$ in $[a,b]$ with the property that

$$\operatorname{lub}\{f(x) : x \in J\} - \operatorname{glb}\{f(x) : x \in J\} < \lambda$$

(assume the contrary and reach a contradiction). We write this (for brevity) as $\operatorname{lub} f - \operatorname{glb} f < \lambda$ on J. Let $I_0 = [a,b]$.

(ii) quad Find a subinterval I_1 of I_0 on which $\operatorname{lub} f - \operatorname{glb} f < 1$;
find a subinterval I_2 of I_1 on which $\operatorname{lub} f - \operatorname{glb} f < 1/2$;
find a subinterval I_3 of I_2 on which $\operatorname{lub} f - \operatorname{glb} f < 1/3$;
etc.

(iii) Let x_0 be a point in the intersection of the I_n. Use the fact that $f(x_0) > 0$ to show that f is bounded below by a positive number in some interval containing x_0.
