Analysis I: Example Sheet 2

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1. The definition $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$, where $s_n = a_1 + \cdots + a_n$, is not the only possible definition for the sum of an infinite series. Another one is

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \frac{s_1 + \cdots + s_n}{n} \qquad (*)$$

(when this limit exists). Let us say that the series is **summable** when the limit $(*)$ exists. Show that if the series $\sum_n a_n$ is convergent then it is summable, and the two limits are the same. Show that if $a_n = (-1)^n$, the corresponding series is summable but not convergent. [You may now ask why we don't always work with summable series.]

2. The theory of **infinite products** $b_1 b_2 \cdots$, or $\prod_{n=1}^{\infty} b_n$, is more subtle than the theory of infinite sums. Here is a start to the theory. We suppose throughout that the a_n are positive numbers. Let

$$s_n = a_1 + \cdots + a_n, \quad p_n = (1 + a_1) \cdots (1 + a_n).$$

Then $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence s_n converges, and (by definition) the infinite product $\prod_{n=1}^{\infty}(1 + a_n)$ converges if and only if the sequence p_n converges. Use the inequality $1 + x \leqslant e^x$ for positive x (and standard properties of the exponential function) to show that $s_n \leqslant p_n \leqslant e^{s_n}$. Hence prove the following

Theorem. *Suppose that $a_n \geqslant 0$ for all n. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\prod_{n=1}^{\infty}(1 + a_n)$ converges.*

According to this result the products

$$\prod_{n=2}^{\infty}\left(1 + \frac{1}{n^2 - 1}\right), \quad \prod_{n=1}^{\infty}\left(1 + \frac{1}{n}\right)$$

converge and diverge, respectively. Give direct proofs of these facts, and find the value of the first of these products.

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3. Suppose that E is a subset of $\mathbb{R}$, and that E has a maximal element e (i.e. $x \leqslant e$ for every x in E). Show that $\operatorname{lub} E = e$.

4. Let $P(z) = (z - a_1)(z - a_2)(z - a_3)(z - a_4)$ and $Q(z) = (z - a_1)(z - a_2)(z - a_3)(z - a_4)(z - a_5)$, where $a_1 < a_2 < a_3 < a_4 < a_5$. Let $A = \{x \in \mathbb{R} : P(x) < 0\}$, and $B = \{x \in \mathbb{R} : Q(x) < 0\}$. Determine whether A and B are (i) bounded above, (ii) bounded below. When one of these bounds exists, find the least upper bound or greatest lower bound as appropriate.

5. Suppose that $z \in \mathbb{C}$, and $R > 0$, and let $E = \{|z - w| : |w| = R\}$. Give $\operatorname{lub} E$ and $\operatorname{glb} E$ and prove your results. [Draw a diagram.]

6. Let P be the parabola given by the equation $y = x^2$ (so $x + iy \in P$ if and only if $y = x^2$). How would you obtain a (reasonably good) numerical estimate of $\operatorname{glb}\{|\zeta - z| : z \in P\}$, where $\zeta = 3 + 7i$? Obtain such an estimate.

7. Suppose that A and B are subsets of $\mathbb{R}$ with the property that if $a \in A$ and $b \in B$ then $a < b$. Prove that $\operatorname{lub} A \leqslant \operatorname{glb} B$.

8. Suppose that A and B are non-empty sets of real numbers, each bounded above, and define

$$A+B=\{a+b : a\in A,\ b\in B\},\quad AB=\{ab : a\in A,\ b\in B\}.$$

Show that $A+B$ is non-empty and bounded above, and that $\operatorname{lub}(A+B)=\operatorname{lub}(A)+\operatorname{lub}(B)$. Show that AB need not be bounded above. If AB is bounded above, is $\operatorname{lub} AB$ necessarily equal to $\operatorname{lub} A\times\operatorname{lub} B$?

9. Let a_n be a real sequence. Show that a_n converges to a if and only if for every pair of real numbers α and β with $\alpha<a<\beta$, there is an n_0 such that $n>n_0$ implies that $\alpha<a_n<\beta$. [*This definition of convergence does not need the concept of distance, only order, and it generalizes easily to give the appropriate definitions of* $x_n\to+\infty$ *and* $x_n\to-\infty$].

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10. Let E be a non-empty subset of $\mathbb{C}$. Suppose that $a_1,\dots a_n$ are complex numbers, and that $f_1,\dots,f_n$ are complex-valued functions that are defined and continuous at every point of E. Show that $a_1f_1+\cdots+a_nf_n$ is continuous at every point of E.
[Question 1, Sheet 1 is an example of a set E, and functions $f_1,f_2,\dots$, each continuous at every point of E, such that the convergent series $\sum_{n=1}^{\infty} f_n(z)$ is not continuous at every point of E].

11. Let $f(z)=\sum_{m=0}^{p}\sum_{n=0}^{q} a_{m,n}x^my^n$, where $z=x+iy$ (with x and y real), and the $a_{i,j}$ are real numbers (thus f is the general real polynomial in the two real variables x and y). Prove that f is continuous on $\mathbb{C}$. [This is easy if you use the appropriate theorems.]

12. In each of the following cases decide whether the function f, which is defined on $\mathbb{R}$ and has $f(0)=0$, is continuous at 0. Justify your answers.
(a) $f(x)=x\sin(1/x)$ when $x\neq 0$;
(b) $f(x)=\sin(1/x)$ when $x\neq 0$;
(c) $f(x)=(1/x)\sin(1/x)$ when $x\neq 0$;
(d) $f(x)=x$ if x is rational, and $f(x)=-x$ if x is irrational.

13. Let $E=\{x : 0\leqslant x<1\}\cup\{x : 2\leqslant x\leqslant 3\}$, and let f be defined on E by

$$f(x)=\begin{cases} x & \text{if } x\in[0,1),\\ x-1 & \text{if } x\in[2,3].\end{cases}$$

Show that f is strictly increasing on E (that is, $x,y\in E$ and $x<y$ implies $f(x)<f(y)$). Is f continuous on E? Let $f(E)=\{f(x) : x\in E\}$, so that f^{-1} exists on $f(E)$. Is f^{-1} continuous on $f(E)$?

14. Suppose that $f:[a,b]\to\mathbb{R}$ is strictly increasing, and let $E=[a,b]$ and $f(E)=\{f(x) : x\in E\}$. Show that
(a) $f^{-1}: f(E)\to E$ is continuous on $f(E)$ *regardless of whether* $f:E\to f(E)$ *is continuous or not*;
(b) $f:E\to f(E)$ is continuous on E if and only if $f(E)$ is an interval.

15. Let $f:\mathbb{R}\to[0,1]$ be defined by

$$f(x)=\begin{cases} 1 & \text{if } x \text{ is an integer},\\ 1/2^n & \text{if } x=p/2^n, \text{ where } p \text{ is an odd integer, and } n=1,2,\dots,\\ 0 & \text{otherwise}.\end{cases}$$

This is sometimes called the **ruler function** (compare the the graph of f with the markings on a ruler in inches). At which points is f (i) continuous (ii) discontinuous?
