# Analysis I: Example Sheet 1, AFB, Lent 2005

**Please send corrections, comments etc. to afb@dpmms.cam.ac.uk**

---

**1.** Show that if $x > 0$ then

$$x + \frac{x}{1+x} + \frac{x}{(1+x)^2} + \frac{x}{(1+x)^3} + \cdots = 1 + x.$$

Show that if $x = 0$ then the series converges to 0 even though $1 + 1/(1+x) + \cdots$ diverges. [This example shows that an infinite sum of continuous functions need not be continuous. Do not worry about a formal definition of continuity at this stage.]

**2.** Discuss the convergence, and (when it converges) the sum, of the series $\sum_{n=0}^{\infty} \left(x^n - x^{n+1}\right)$.

**3.** Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}, \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}, \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)(n+3)}.$$

By considering the expression $1/n - 1/(n+1)$ show that the first series converges to 1. By considering partial fractions, find the value of the other two series.

**4.** Consider the series $S = a_1 + a_2 + a_3 + \cdots$ and $S^* = b_1 + b_2 + b_3 + \cdots$, where $b_1 = a_1 + a_2$, $b_2 = a_3 + a_4$, $b_3 = a_5 + a_6$, and so on. Show that if $S$ converges then so does $S^*$. Give an example in which $S^*$ converges yet $S$ diverges. [This example shows (essentially) that we can insert brackets into a convergent series without changing its sum. It also shows that *we cannot always remove brackets and expect the same answer* !]

**5.** Determine whether each of the following series converges or diverges. If the series converges obtain an upper bound for its sum.

$$\sum_{n=1}^{\infty} \frac{2n}{5n^3 - n + 6}, \quad \sum_{n=2}^{\infty} \frac{n^{n/2} - 1}{n^n - 1}, \quad \sum_{n=1}^{\infty} \frac{n^2}{3^n}, \quad \sum_{n=1}^{\infty} (|\sin n| + |\cos n|).$$

**6.** Suppose that $x_1, x_2, \ldots$ are positive, and let

$$S_1 = \sum_{n=1}^{\infty} \sqrt{x_n}, \quad S_2 = \sum_{n=1}^{\infty} \sqrt{\frac{x_n}{1+x_n}}, \quad S_3 = \sum_{n=1}^{\infty} \sqrt{x_n x_{n+1}}.$$

Show that if $S_1$ converges then so do $S_2$ and $S_3$. If $S_2$ converges does $S_1$ necessarily converge? If $S_3$ converges does $S_1$ necessarily converge?

**7.** For $n = 1, 2, \ldots$, let

$$a_n = \frac{1}{\sqrt{n}} + \frac{(-1)^{n-1}}{n}.$$

Show that each $a_n$ is positive, and that $a_n \to 0$ as $n \to \infty$. Show also that $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ *diverges.* [This shows that, in the Alternating Series Test, it is essential to know that the moduli of the terms decrease as $n$ increases.]

**8.** Let

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}.$$

Show that the sequence $a_n$ is increasing and bounded above. Deduce that $a_n$ converges, say to $A$, where $A > \frac{1}{2}$. In particular, $a_n$ does not converge to 0 even though it may appear that each term in the expression for $a_n$ converges to $0$.

**9.** Suppose that $k > 1$ and also that the real numbers $a_0, a_1, \ldots$ satisfy

$$a_{n+2} - (k + k^{-1})a_{n+1} + a_n = 0, \quad n \geqslant 0.$$

Given that $a_0 = 1$ find all values of $a_1$ such that the sequence $a_n$ converges.

**10.** We are given $x_1$, and we define the sequence $x_n$ inductively by the recurrence relation

$$x_{n+1} = x_n^2 - 2x_n + 2, \quad n = 1, 2, \ldots.$$

Make a conjecture about the behaviour of $x_n$ as $n \to \infty$ in each of the cases (i) $1 < x_1 < 2$, (ii) $x_1 = 2$, (iii) $x_1 > 2$. Now prove your conjectures.
[Let $f(x) = x^2 - 2x + 2$; then $x_{n+1} = f(x_n)$, so it is clear that it might help to sketch the graph of $y = f(x)$. You might also predict the answers by evaluating $x_1, \ldots, x_{100}$, say, on a computer for selected values of $x_1$.]

**11. This question is important.** Let

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots, \quad S' = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots.$$

It is important to understand how $S'$ is constructed. We add the first two terms of $1, 1/3, 1/5, 1/7, 1/9, \ldots$ and subtract the first term of $1/2, 1/4, 1/6, \ldots$. We then add the next two terms of $1, 1/3, 1/5, 1/7, 1/9, \ldots$, and subtract the next term of $1/2, 1/4, 1/6, \ldots$, and so on in the obvious way. Notice that *$S$ and $S'$ have the same terms, but they are added together in a different order.* Notice also that the infinite series for $S$ converges, so that $S$ is a real number. We do not yet know whether the series for $S'$ converges or diverges.
Let $S_n$ and $S'_n$ be the sum of the first $n$ terms of $S$ and $S'$, respectively, and let

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Prove (for example, by induction) that

$$S_{2n} = H_{2n} - H_n, \quad S'_{3n} = H_{4n} - \tfrac{1}{2}H_{2n} - \tfrac{1}{2}H_n.$$

Deduce that $S'_{3n} = S_{4n} + \frac{1}{2}S_{2n}$, and hence that the series for $S'$ also converges. Deduce also that $2S' = 3S$, and hence that $S \neq S'$ (you have to show here that $S \neq 0$). In conclusion, *despite the fact that the terms in $S$ and $S'$ are the same, and that both series converge, we do have $S \neq S'$.*

**12.** Let $\theta$ be a real number. Discuss the behaviour of the sequence $\cos(n\theta)$ as $n \to \infty$. You are not required to give proofs, but you are required to collect evidence, speculate, and so on. You may wish to compare this with the sequence $e^{in\theta}$ of complex numbers that lie on the unit circle.

---