Brief Notes on Analysis I: Sheet 1

We use the standard notation:
$\mathbb{N}$ is $\{1, 2, 3, \ldots\}$ (the set of positive integers),
$\mathbb{Z}$ is $\{0, 1, -1, 2, -2, \ldots\}$ (the set of all integers),
$\mathbb{Q}$ is the set of rational numbers,
$\mathbb{R}$ is the set of real numbers,
$\mathbb{C}$ is the set of complex numbers.

Sequences
Informally, a sequence is a list $s_1, s_2, \ldots$ of numbers. This is meant to imply that if we choose an integer n then we know what s_n is; thus, formally, a sequence is a function from $\{1, 2, 3, \ldots\}$ to $\mathbb{R}$ (a real sequence) or to $\mathbb{C}$ (a complex sequence). Obviously we can also consider sequences such as $x_{-1}, x_0, x_1, \ldots$, or even doubly infinite sequences $\ldots, z_{-1}, z_0.z_1, \ldots$.

We want to create a formal definition of the **convergence** of a sequence to a limit, and we want to be sure that (for example) $1/n \to 0$ as $n \to \infty$.

Informally, the sequence a_n converges to a if, given an agreed 'error', all but a finite number of the terms a_n are within this agreed error of a. It is customary to denote the error by ε, so we arrive at the following formal definition (for a complex sequence):
the sequence a_n converges to a if, given any positive ε, there is an integer n_0 (which will depend on ε) such that if $n > n_0$ then $|a_n - a| < \varepsilon$. If a_n converges to a we write $a_n \to a$, and also $\lim_{n\to\infty} a_n = a$. We call a the **limit** of the sequence a_n. Any sequence can have at most one limit.

The existence and values of the limits of sequences interacts with the algebra of real and complex numbers in the natural way:
if $a_n \to a$ and $b_n \to b$ then (with appropriate modifications in the case of the quotient)

$$\lambda a_n + \mu b_n \to \lambda a + \mu b, \quad a_n b_n \to ab, i \quad a_n/b_n \to a/b.$$

To make further progess we need something more substantial. We now introduce the following **AXIOM: if a_n is a real sequence such that (i) $a_1 \leqslant a_2 \leqslant \cdots$, and (ii) for some M, and all n, $a_n \leqslant M$, then the sequence a_n converges to some a.**
It is easy to see that necessarily, $a \leqslant M$ (and possibly $a < M$).

The following terminology is helpful:
(1) the real sequence $x_1, x_2, \ldots$ is **increasing** if $x_n \leqslant x_{n+1}$ for all n;
(2) the real sequence $x_1, x_2, \ldots$ is **decreasing** if $x_n \geqslant x_{n+1}$ for all n;
(3) a real sequence is **monotonic** if it is either increasing or decreasing;
(4) the real sequence $x_1, x_2, \ldots$ is **strictly increasing** if $x_n < x_{n+1}$ for all n;
(5) the real sequence $x_1, x_2, \ldots$ is **strictly decreasing** if $x_n > x_{n+1}$ for all n;
(6) the real sequence $x_1, x_2, \ldots$ is **bounded above** if there is an M such that $x_n \leqslant M$ for all n;
(7) the real sequence $x_1, x_2, \ldots$ is **bounded below** if there is an M such that $x_n \geqslant M$ for all n;
(8) the COMPLEX sequence $z_1, z_2, \ldots$ is **bounded** if there is an M such that $|z_n| \leqslant M$ for all n.
Our AXIOM now states that **any real monotonic bounded sequence converges**.

Infinite series
An infinite series is an expression of either of the forms

$$\sum_{n=1}^{\infty} a_n, \quad a_1 + a_2 + \cdots.$$

We write $s_n = a_1 + a_2 + \cdots + a_n$; these are the **partial sums** of the infinite series. The series $\sum_{n=1}^{\infty} a_n$ **converges** if and only if the sequence s_n converges, and then we give $\sum_{n=1}^{\infty} a_n$ the value $\lim_{n\to\infty} s_n$. If a series does not converge it is said to **diverge**.

Note that we can add, or delete, any FINITE number of terms to, or from, the series without affecting its convergence/divergence. We would, of course, affect the actual value of the series (when it converges) but this is less important.

It is clear (from the corresponding result for sequences) that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge then so does $\sum_{n=1}^{\infty}(\lambda a_n + \mu b_n)$ and

$$\sum_{n=1}^{\infty}(\lambda a_n + \mu b_n) = \lambda \sum_{n=1}^{\infty} a_n + \mu \sum_{n=1}^{\infty} b_n.$$

We need tests for convergence, and the most powerful test of all comes directly from our AXIOM:
if $a_n \geqslant 0$ for all n, and if there exists M such that $a_1 + \cdots + a_n \leqslant M$ for all n, then $\sum_{n=1}^{\infty} a_n$ converges.
This leads directly to

The Comparison Test: *if $0 \leqslant a_n \leqslant b_b$ for all n, and if $\sum_{n=1}^{\infty} b_n$ converges then so does $\sum_{n=1}^{\infty} a_n$.*

Proof Suppose that $\sum_n b_n$ converges to B, Then for all n, $b_1 + \cdots + b_n \leqslant B$ (**prove this**). Thus for all n, $a_1 + \cdots + a_n \leqslant B$ so by the result above, $\sum_n a_n$ converges.

The series $\sum_n z_n$ is said to **absolutely convergent** if $\sum_n |z_n|$ is convergent.

Theorem. *If $\sum_n z_n$ is absolutely convergent then it is convergent.*

Proof Write $z_n = x_n + iy_n$. As $0 \leqslant |x_n| \pm x_n \leqslant 2|x_n| \leqslant 2|z_n|$, we see that the two series $\sum_n(|x_n| + x_n)$ and $\sum_n(|x_n| - x_n)$ both converge. Thus so does their difference, namely $\sum_n 2x_n$, and hence so too does $\sum x_n$. A similar argument shows that $\sum_n y_n$ converges; hence so does $\sum_n(x_n + iy_n)$. **Despite this simple proof, this is a fundamental and important result; it applies to complex sequences.**

If we combine some of these tests we have the following powerful test for complex series:
Suppose that for some M, and all n, $|z_1| + \cdots + |z_n| \leqslant M$. Then $\sum_n z_n$ converges.

The **Ratio test**: suppose that $z_n \neq 0$ for any n.
(i) If there is some k with $0 < k < 1$ such that for all n, $|z_{n+1}/z_n| \leqslant k$, then $\sum_n |z_n|$ converges
(ii) If $|z_{n+1}/z_n| \geqslant 1$ then $\sum_n |z_n|$ diverges.

The proof of (ii) is easy. As $|z_n| \geqslant |z_{n-1}| \geqslant \cdots \geqslant |z_1| > 0$ we see that the sequence z_n does NOT converge to 0. On the other hand if the series $\sum z_n$ converges, then $s_n = z_1 + \cdots + z_n \to s$, say and then $z_n = s_n - s_{n-1} \to s - s = 0$.

If a series $\sum_n z_n$ converges but $\sum_n |z_n|$ diverges, we say that the series is **conditionally convergent**. Tests for these series are usually much more delicate, but there is one elementary test.

The Alternating Series Test. *Suppose that $a_1 \geqslant a_2 \geqslant \cdots \geqslant 0$ and that $a_n \to 0$ as $n \to \infty$. Then the series $a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$ converges.*

Here are some limits/series that you should be familiar with.

- $\sum_n z^k$ is convergent if $|z| < 1$, and divergent if $|z| \geqslant 1$;
- $\sum_n 1/n^k$ is convergent if $k > 1$, and divergent if $k \leqslant 1$;
- $1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \cdots$ is convergent, but not absolutely convergent.
- The series $\exp z = 1 + z + z^2/2! + z^3/3! + \cdots$ is absolutely convergent for every complex number z. Thus $\exp z$ is now a function defined everywhere on $\mathbb{C}$.
- $n^{1/n} \to 1$ as $n \to \infty$; thus if $1 \leqslant a_n \leqslant n$, then $a_n^{1/n} \to 1$ as $n \to \infty$. In particular, for any positive a, $a^{1/n} \to 1$ as $n \to \infty$.