SYMPLECTIC GEOMETRY EXAMPLES 3

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`. Most of the exercises are taken from the text by A. Cannas da Silva that we are following. The two questions with a * are intended for marking.

1*. Let (V, Ω) be a symplectic vector space. Recall that a complex structure J on V is a linear map $J : V \to V$ with $J^2 = -\mathrm{id}$. The complex structure J is said to be *compatible* with Ω if $J^*\Omega = \Omega$ and $\Omega(u, Ju) > 0$ for all $u \neq 0$. Show that there is always a compatible complex structure J on (V, Ω).

2. Let M be a compact k-dimensional submanifold of $\mathbb{R}^n$. Suppose that at each $p \in X$ we are given a linear isomorphism $L_p : \mathbb{R}^n \to \mathbb{R}^n$ such that $L_p|_{T_pX} = \mathrm{id}$ and L_p depends smoothly on p. Show that there exists an embedding $h : N \to \mathbb{R}^n$ of some neighbourhood N of X such that $h|_X = \mathrm{id}$ and $dh_p = L_p$ for all $p \in X$.

3. Let (M, H) be a contact manifold with a contact form α. Let $N = M \times \mathbb{R}$ and let $\pi : N \to M$ be the projection $\pi(x, s) = x$. Show that $\omega = d(e^s \pi^*\alpha)$ is a symplectic form on N.

4*. Let (M, H) be a contact manifold and $p \in M$. Show that there exists a coordinate system $(U, x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ centered at p such that on U

$$\alpha = dz + \sum_{i=1}^{n} x_i dy_i$$

is a local contact form for H.

5. (Gray stability) Let M be a compact manifold without boundary. Assume that α_t, $t \in [0, 1]$, is a smooth family of contact forms on M. Show that there exists an isotopy $\rho_t : M \to M$ and a family of smooth nowhere vanishing functions $u_t : M \to \mathbb{R}$, $t \in [0, 1]$, such that $\rho_t^*\alpha_t = u_t\,\alpha_0$ for all $t \in [0, 1]$ (hint: search for a time dependent vector field which belongs to the kernel of α_t).

6. The manifold of contact elements of an n-dimensional manifold X is

$$\mathcal{C} = \{(x, \chi_x) :\ x \in X \text{ and } \chi_x \text{ is a hyperplane in } T_xX\}.$$

On the other hand, the projectivization of the cotangent bundle of X is

$$\mathbb{P}^*X = (T^*X \setminus \text{zero section})/\sim$$

where $(x, \xi) \sim (x, \xi')$ whenever $\xi = \lambda\xi'$ for some $\lambda \in \mathbb{R} \setminus \{0\}$ (here $x \in X$ and $\xi, \xi' \in T_x^*X \setminus \{0\}$). We denote the elements of $\mathbb{P}^*X$ by $(x, [\xi])$, where $[\xi]$ is the equivalence class of ξ. Show that $\mathcal{C}$ is naturally isomorphic to $\mathbb{P}^*X$ as a bundle over

X, in other words, exhibit a diffeomorphism $\varphi : \mathcal{C} \to \mathbb{P}^* X$ such that $\pi_{\mathbb{P}^* X} \circ \varphi = \pi_{\mathcal{C}}$, where $\pi_{\mathbb{P}^* X}(x, [\xi]) = x$ and $\pi_{\mathcal{C}}(x, \chi_x) = x$.

The manifold $\mathcal{C}$ carries a natural field of hyperplanes $\mathcal{H}$, where

$$\mathcal{H}_{(x,\chi_x)} = (d\pi_{\mathcal{C}})^{-1}_{(x,\chi_x)}(\chi_x).$$

Via φ, $\mathcal{H}$ induces a field of hyperplanes $\mathbb{H}$ on $\mathbb{P}^* X$. Describe $\mathbb{H}$.

7. Prove that $(\mathbb{P}^* X, \mathbb{H})$ is a contact manifold (and hence so is $(\mathcal{C}, \mathcal{H})$).

8. What is the symplectization of $\mathcal{C}$? What is the manifold $\mathcal{C}$ when $X = \mathbb{R}^3$ and when $X = S^1 \times S^1$?

9. Let $X = \mathbb{R}^2$, $\sigma = dx \wedge dy$ and let $H : T^* X \to \mathbb{R}$ be given by $H(x, \xi) = |\xi|^2/2$. Describe the orbits of the Hamiltonian vector field of H with respect to the twisted symplectic form ω_σ on $T^* X$. What happens to the orbits if we consider instead $\omega_{-\sigma}$?

10. Let (V, Ω) be a symplectic vector space and let $\mathcal{J}(V, \Omega)$ denote the set of all compatible complex structures. Show that $\mathcal{J}(V, \Omega)$ is contractible.