SYMPLECTIC GEOMETRY EXAMPLES 2

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`. Most of the exercises are taken from the text by A. Cannas da Silva that we are following. The two questions with a * are intended for marking.

1*. The standard symplectic form on the unit 2-sphere S^2 is given by

$$\omega_u(v,w) = \langle u, v \times w \rangle,$$

where $u \in S^2$, $v, w \in T_u S^2$ are vectors in $\mathbb{R}^3$, $\times$ is the cross product and $\langle \cdot, \cdot \rangle$ is the standard inner product. Consider cylindrical polar coordinates (θ, z) on S^2 where $\theta \in [0, 2\pi)$ and $z \in [-1, 1]$. Show that in these coordinates $\omega = d\theta \wedge dz$.

2. Let M and X be manifolds with $\dim X < \dim M$ and let $f : X \to M$ be a smooth map. Recall that f is an immersion if $df_p : T_pX \to T_{f(p)}M$ is injective for all $p \in X$. Show that the following are equivalent:

(1) f is an immersion that is a homeomorphism onto its image (with the relative topology) with $f(X)$ closed;
(2) f is a proper injective immersion (recall that f is proper if the preimage of any compact set is compact).

3. Let M be a manifold and ν_t a time-dependent vector field generated by a smooth isotopy $\rho_t : M \to M$, $t \in \mathbb{R}$. Show Cartan's formula

$$\mathcal{L}_{\nu_t}\omega = \iota_{\nu_t} d\omega + d\iota_{\nu_t}\omega,$$

where $\omega \in \Omega^k(M)$ by reducing to the autonomous case and using Cartan's formula for time-independent vector fields.

4*. Let X be a manifold and consider the cotangent bundle $\pi : T^*X \to X$ equipped with its canonical symplectic form $\omega = -d\alpha$, where α is the Liouville 1-form. Let σ be a closed 2-form on X and define

$$\omega_\sigma := \omega + \pi^*\sigma.$$

Show that ω_σ is exact if and only if σ is exact.

Are ω_σ and $\omega_{-\sigma}$ deformation equivalent? Are they isotopic? Suppose there is a diffeomorphism $f : X \to X$ such that $f^*\sigma = -\sigma$, show that ω_σ and $\omega_{-\sigma}$ are symplectomorphic.

5. Let X be a k-dimensional submanifold of an n-dimensional manifold M. The normal space to X at $x \in X$ is the quotient space $N_xX = T_xM/T_xX$ and the normal bundle of X in M is the vector bundle NX over X whose fibre at x is N_xX. Prove that NX is indeed a vector bundle. If M is $\mathbb{R}^n$, show that N_xX can be identified with the usual normal space to X in $\mathbb{R}^n$, that is, the orthogonal complement in $\mathbb{R}^n$ of T_xX.

6. Let $X \subset \mathbb{R}^n$ be a compact k-dimensional submanifold. Given $\varepsilon > 0$, let U_ε be the set of points in $\mathbb{R}^n$ which are at distance less than ε from X. Let $NX_\varepsilon := \{(x, v) \in NX : |v| < \varepsilon\}$ and let $f : NX \to \mathbb{R}^n$ be the map $f(x, v) = x + v$. Show that for ε sufficiently small, f maps NX_ε diffeomorphically onto U_ε.

7. Suppose that the manifold in the previous exercise is not compact. Prove that the assertion about f is still true provided we replace ε by a continuous function $\varepsilon : X \to (0, \infty)$ which tends to zero fast enough as x tends to infinity.

8. Prove the Darboux theorem in the 2-dimensional case, using the fact that every nonvanishing 1-form on a surface can be written locally as $f dg$ for suitable functions f and g.

9. Let M be an orientable surface. Show that convex combinations of two area forms ω_0 and ω_1 that induce the same orientation are symplectic. Show that this fails in dimension four: find two symplectic forms in $\mathbb{R}^4$ that induce the same orientation, yet some convex combination of which is degenerate. Find a path of symplectic forms that connect them.

10. Let ω_0 and ω_1 be area forms on a compact surface M such that their cohomology classes agree, i.e. $[\omega_0] = [\omega_1]$. Show that that there exists a smooth 1-parameter family of diffeomorphisms $\varphi_t : M \to M$ such that $\varphi_1^*\omega_0 = \omega_1$, $\varphi_0 = \mathrm{id}$ and $\varphi_t^*\omega_0$ is symplectic for all $t \in [0, 1]$.