SYMPLECTIC GEOMETRY EXAMPLES 1

G.P. Paternain Michaelmas 2021

Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`. Most of the exercises are taken from the text by A. Cannas da Silva that we are following. The two questions with a * are intended for marking.

1*. Given a linear subspace Y of a symplectic vector space (V, Ω), its *symplectic orthogonal* Y^Ω is the linear subspace defined by

$$Y^\Omega := \{v \in V : \Omega(v, u) = 0 \text{ for all } u \in Y\}.$$

(a) Show that $\dim Y + \dim Y^\Omega = \dim V$.
(b) Show that $(Y^\Omega)^\Omega = Y$.
(c) Show that, if Y and W are subspaces, then $Y \subseteq W$ iff $W^\Omega \subseteq Y^\Omega$.

2. We call Y *isotropic* when $Y \subseteq Y^\Omega$. Show that, if Y is isotropic, then $\dim Y \leq \frac{1}{2}\dim V$. We call Y *coisotropic* when $Y^\Omega \subseteq Y$. Check that every codimension 1 subspace is coisotropic.

3. An isotropic subspace Y of (V, Ω) is called *Lagrangian* when $\dim Y = \frac{1}{2}\dim V$. Check that Y is Lagrangian iff Y is both isotropic and coisotropic iff $Y = Y^\Omega$. Show that, if Y is a Lagrangian subspace, then any basis $\{e_1, \dots, e_n\}$ of Y can be extended to a symplectic basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ of V.

4. Let E be a (finite dimensional) real vector space. Consider the bilinear form on $V = E \oplus E^*$ given by

$$\Omega(u + \alpha, v + \beta) = \beta(u) - \alpha(v).$$

Is Ω symplectic?

5*. Let V be a $2n$-dimensional real vector space and $\Omega \in \Lambda^2(V^*)$ (the set of skew-symmetric bilinear forms on V). Show that Ω is symplectic iff the nth exterior power $\Omega^n = \underbrace{\Omega \wedge \cdots \wedge \Omega}_{n}$ is not zero.

6. Show that if $n > 1$ there are no symplectic structures on the sphere S^{2n}.

7. Let (M, ω) be a symplectic manifold, and let α be a 1-form such that $\omega = -d\alpha$. Show that there exists a unique vector field ν such that its interior product with ω is α, i.e. $\iota_\nu \omega = -\alpha$. Prove that, if g is a symplectomorphism which preserves α (that is, $g^*\alpha = \alpha$), then g commutes with the flow of ν.

Let X be an arbitrary n-dimensional manifold and let $M = T^*X$. Let $(x_1, \dots, x_n)$ be coordinates defined on a neighbourhood U, and let $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ be corresponding coordinates on T^*U. Show that when α is the canonical (or Liouville/tautological) 1-form on M, the vector field ν in the previous exercise is $\sum \xi_i \frac{\partial}{\partial \xi_i}$. Moreover, show that that the flow ϕ_t of ν is given by $\phi_t(x,\xi) = (x, e^t\xi)$.

8. Let $M = T^*X$ and α the canonical 1-form. Show that if g is a symplectomorphism of M which preserves α, then $g(x,\xi) = (y,\eta)$ implies $g(x,\lambda\xi) = (y,\lambda\eta)$ for all $(x,\xi) \in M$ and $\lambda \in [0,\infty)$. Conclude that g preserves the cotangent fibration, i.e. show that there exists a diffeomorphism $f : X \to X$ such that $\pi \circ g = f \circ \pi$, where $\pi : M \to X$ is the projection map $\pi(x,\xi) = x$. Finally prove that $g = f_\#$, where $f_\#$ is the symplectomorphism of M lifting f.

9. Let $M = T^*X$ and α the canonical 1-form. Let θ be a 1-form on X. Define $\tau_\theta : M \to M$ by setting

$$\tau_\theta(x,\xi) := (x, \xi + \theta_x).$$

Compute $\tau_\theta^*\alpha$. Give a necessary and sufficient condition on θ so that τ_θ is a symplectomorphism. Give an example of a symplectomorphism of M which does not preserve the canonical 1-form α.

10. Let X be a manifold and consider the cotangent bundle $\pi : T^*X \to X$ equipped with its canonical symplectic form $\omega = -d\alpha$, where α is the Liouville 1-form. Let σ be a closed 2-form on X and define

$$\omega_\sigma := \omega + \pi^*\sigma.$$

Show that ω_σ is a symplectic form.

Let θ be a 1-form on X which we also regard as a section $\theta : X \to T^*X$. Show that $\theta(X)$ is a Lagrangian submanifold of (T^*X, ω_σ) if and only if $\sigma = d\theta$. Conclude that if the cohomology class of σ is not zero, then there are no Lagrangian submanifolds L in (T^*X, ω_σ) for which $\pi|_L : L \to X$ is a diffeomorphism. Assume that σ is exact, is it true that (T^*X, ω) and (T^*X, ω_σ) are symplectomorphic?