EXAMPLE SHEET 3

1. Let U be a flow domain and let $\Phi : U \to X$ be a smooth map satisfying $\Phi^s \circ \Phi^t = \Phi^{s+t}$ wherever this makes sense. Show that Φ is a local flow of a vector field v on X, which you should define.

2.† (a) Compute the coordinate expression for the Lie derivative of a 1-form S_a and of a tensor $T_a{}^b$, along a vector field v.
   (b) Write down the flow of the vector field $\mathsf{v} = x\,\partial_y - y\,\partial_x$ on $\mathbb{R}^2$, and hence compute the Lie derivative $\mathcal{L}_{\mathsf{v}}$ of the 1-form $x\,\mathrm{d}y$ directly from the definition.

3. Prove Cartan's magic formula $\mathcal{L}_{\mathsf{v}}\alpha = \mathrm{d}(\iota_{\mathsf{v}}\alpha) + \iota_{\mathsf{v}}(\mathrm{d}\alpha)$, for a vector field v and r-form α on a manifold X, as follows. Let $\Phi : U \to X$ be a local flow of v, and consider the map $F : U \to \mathbb{R} \times X$ given by $F(t, p) = (t, \Phi^t(p))$, viewed as a diffeomorphism onto its image V.
   (a) Prove Cartan's formula for an r-form β on U, and for the vector field ∂_t representing translation in the $\mathbb{R}$ direction, by direct calculation.
   (b) Use diffeomorphism-invariance under F to obtain the result for $\mathrm{pr}_2^*\,\alpha$ and $\partial_t \oplus \mathsf{v}$ on V.
   (c) Deduce the result for α and v on X.

4. Let v be a vector field on X with local flow Φ.
   (a) Show that, for any tensor T, if $\mathcal{L}_{\mathsf{v}}T = 0$ then $(\Phi^t)^*T = T$ wherever this makes sense.
   (b) Let w be another vector field, with local flow Ψ. For small t and u, show that $\Phi^{-t} \circ \Psi^u \circ \Phi^t$ is the time-u flow of $(\Phi^t)^*\mathsf{w}$, and deduce that if $[\mathsf{v}, \mathsf{w}] = 0$ then Φ^t and Ψ^u commute.

5. Let X and Y be manifolds of dimensions n and m, and suppose $F : X \to Y$ is a submersion at p. Construct an open neighbourhood U of p and a smooth map $G : U \to \mathbb{R}^{n-m}$ such that $(F|_U, G) : U \to Y \times \mathbb{R}^{n-m}$ is a local diffeomorphism at p. Deduce that there exist local coordinates on X and Y about p and $F(p)$ in which F is given by projection onto the first m components.

6. Show that there is no surjective smooth map $f : X \to Y$ if $\dim X < \dim Y$.

7. Let $\pi : X \to Y$ be a submersion, and let D be a k-plane distribution on X transverse to the fibres, where $k = \dim X - \dim Y$. A curve in X is *horizontal* if it is everywhere tangent to D. Given a point p in X and a curve $\overline{\gamma} : [0, 1] \to Y$ with $\overline{\gamma}(0) = \pi(p)$, show that for small $\varepsilon > 0$ there is a unique horizontal curve $\gamma : [0, \varepsilon] \to X$ satisfying $\gamma(0) = p$ and $\pi \circ \gamma = \overline{\gamma}$. If we can take $\epsilon = 1$ then $\gamma(1)$ is the *parallel transport* of p along $\overline{\gamma}$. Show that D is integrable iff for all p there exists a neighbourhood U of p in $\pi^{-1}(\pi(p))$ and a neighbourhood V of $\pi(p)$ in Y such that for all q in U and all curves $\overline{\gamma}$ in V with $\overline{\gamma}(0) = \pi(q)$ the parallel transport of q along $\overline{\gamma}$ exists and depends only on q and $\overline{\gamma}(1)$, not on the whole curve $\overline{\gamma}$.

8.† (a) By considering the map $F : \mathrm{GL}(n, \mathbb{R}) \to \{\text{symmetric matrices}\}$ given by $F(A) = A^T A$, show that $\mathrm{O}(n)$ is an embedded Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$. Identify $\mathfrak{o}(n)$ as a subspace of $\mathfrak{gl}(n, \mathbb{R})$.
   (b) Show that $\mathrm{SU}(n)$ is a Lie subgroup of $\mathrm{GL}(n, \mathbb{C})$ and similarly identify its Lie algebra.

9. For a Lie group G compute the derivatives $D_{(e,e)}m : \mathfrak{g} \oplus \mathfrak{g}$ and $D_e i : \mathfrak{g} \to \mathfrak{g}$ of the multiplication and inversion maps m and i at the identity. Show that i^* exchanges left-invariant and right-invariant vector fields, and deduce that the bracket operation on $\mathfrak{g}$ defined using right-invariant (rather than left-invariant) vector fields differs from the usual one by a sign.

10. Let $F : H \to G$ be a morphism of Lie groups, i.e. a smooth map which is a group homomorphism. Show that $F(\exp_H(\xi)) = \exp_G(D_e F(\xi))$ for all $\xi \in \mathfrak{h}$, and deduce that $D_e F$ is a Lie algebra homomorphism, i.e. a linear map which respects the bracket operation. This shows, in particular, that if H is an embedded Lie subgroup of G then the exponential map and bracket on $\mathfrak{h}$ are the restrictions of those on $\mathfrak{g}$.

11.* Let G be a Lie group. Given an embedded Lie subgroup H, show that its left cosets induce a foliation of G, with tangent distribution $\{\mathsf{l}_\xi : \xi \in \mathfrak{h}\}$. Given instead a subspace $\mathfrak{h}$ of $\mathfrak{g}$, show that the distribution $\{\mathsf{l}_\xi : \xi \in \mathfrak{h}\}$ on G arises from a foliation iff $\mathfrak{h}$ is actually a Lie subalgebra of $\mathfrak{g}$, i.e. $\mathfrak{h}$ is closed under the Lie bracket on $\mathfrak{g}$. If this holds, must $\mathfrak{h}$ be the Lie algebra of an embedded subgroup of G?

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