

STOCHASTIC FINANCIAL MODELS

Example Sheet 2

1. Consider a multi-period asset price model $(S_n^0, S_n)_{0 \leq n \leq T}$ with numéraire S^0 and d risky assets $S_1, \dots, S^d$. Write X_n for the discounted price vector S_n/S_n^0 . Let $\tilde{\mathbb{P}}$ be an equivalent probability measure such that $\tilde{\mathbb{E}}(|X_n|) < \infty$ for all n . Show that the following are equivalent:

- (a) $\tilde{\mathbb{P}}$ is an equivalent martingale measure,
- (b) for any previsible self-financing portfolio $(\theta_n^0, \theta_n)_{1 \leq n \leq T}$ with $(\theta_n)_{1 \leq n \leq T}$ bounded, the discounted final value V_T satisfies $\tilde{\mathbb{E}}(V_T) = 0$.

Hence show that if there exists any equivalent martingale measure then the model has no bounded arbitrage.

2. Consider a multi-period binomial model $(S_n)_{0 \leq n \leq T}$ with parameters $-1 < a < b$ and interest rate $r \in (a, b)$. Let $EC(S_0, K, T)$ denote the no arbitrage time-0 price of a European call $C = (S_T - K)^+$ with strike K and maturity T . Assume that $K < S_0(1 + b)^T$. Show that $EC(S_0, K, T)$ is increasing in S_0 , decreasing and convex in K and non-decreasing in T .

3. Let $(S_n)_{0 \leq n \leq T}$ be a binomial model with parameters $a < b$ and interest rate $r \in (a, b)$. Assume that $(1 + a)(1 + b) = 1$ and set $p^* = (r - a)/(b - a)$. Show that the fair price at time 0 for the contingent claim

$$C = \min_{0 \leq n \leq T} S_n$$

is given by

$$V_0 = \sum_{k=0}^T p^{*T-k} (1 - p^*)^k \sum_{\substack{m=0 \\ m \geq 2k-T}}^k \left(\binom{T}{k-m} - \binom{T}{k-m-1} \right) \frac{S_0(1+a)^m}{(1+r)^T}.$$

4. Let $(\bar{S}_n)_{0 \leq n \leq T}$ be an asset price model with constant numéraire $S_n^0 = 1$ and d independent risky assets $S_n^1, \dots, S_n^d$, where $S_n^i = e^{W_n^i}$ with $W_0^i = 0$ and $W_n^i = Y_1^i + \dots + Y_n^i$, where, for $i = 1, \dots, d$, $Y_1^i, Y_2^i, \dots$ are independent Gaussian random variables having the same mean μ_i and variance $\sigma_i^2 > 0$. Show that $\mathbb{E}(e^{uY_n^i}) = e^{u\mu_i + u^2\sigma_i^2/2}$ for all $u \in \mathbb{R}$. Deduce that $(\bar{S}_n)_{0 \leq n \leq T}$ has no arbitrage.

5. Let $(B_t)_{t \geq 0}$ be a Brownian motion and let $c \in (0, \infty)$ and $s \in [0, \infty)$. Show that the following processes are also Brownian motions: (a) $(-B_t)_{t \geq 0}$, (b) $(c^{-1}B_{c^2t})_{t \geq 0}$, (c) $(B_{s+t} - B_s)_{t \geq 0}$.

6. Let $(B_t)_{t \geq 0}$ be a Brownian motion and let $\theta \in \mathbb{R}$. Define $Q_t = B_t^2 - t$ and $Z_t = e^{\theta B_t - \theta^2 t/2}$. Show directly that the processes $(Q_t)_{t \geq 0}$ and $(Z_t)_{t \geq 0}$ are continuous martingales, in a suitable filtration, to be specified.

7. Let $(B_t)_{t \geq 0}$ be a Brownian motion and let $a \geq 0$. Set $T_a = \inf\{t \geq 0 : B_t = a\}$. Use the optional stopping theorem to show that, for all $\lambda \geq 0$,

$$\mathbb{E}(e^{-\lambda T_a}) = e^{-a\sqrt{2\lambda}}.$$

8. Let $(B_t)_{t \geq 0}$ be a Brownian motion and set

$$M_t = \sup_{0 \leq s \leq t} B_s, \quad Z_t = M_t - B_t.$$

- (a) Show that (M_t, Z_t) has the same distribution as $\sqrt{t}(M_1, Z_1)$.
- (b) Use the reflection principle to find the joint density of (B_1, M_1) .
- (c) Find the joint density of (M_1, Z_1) .
- (d) Show that, for $u \neq v$,

$$\mathbb{E}(e^{uM_t + vZ_t}) = \frac{u\Psi(\sqrt{t}u) - v\Psi(\sqrt{t}v)}{u - v}, \quad \Psi(u) = 2e^{u^2/2}\Phi(u).$$

Suppose now, more generally, that $(B_t)_{t \geq 0}$ is a Brownian motion with drift $c \in \mathbb{R}$. Thus, $B_t = W_t + ct$ for some Brownian motion $(W_t)_{t \geq 0}$.

- (e) Show that, for $u \neq -2c$,

$$\mathbb{E}(e^{uM_t}) = 2 \left(\frac{c + u}{2c + u} \right) \Phi((c + u)\sqrt{t})e^{cut + u^2t/2} + 2 \left(\frac{c}{2c + u} \right) \Phi(-c\sqrt{t}).$$