

STOCHASTIC FINANCIAL MODELS

Example Sheet 1

1. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = [0, 1)$, $\mathcal{F}$ is the Borel σ -algebra on $[0, 1)$ and $\mathbb{P}$ is Lebesgue measure. Each $x \in \Omega$ has a binary expansion $x = 0.x_1x_2x_3\ldots$ where $x_n \in \{0, 1\}$ for all n . We make this expansion unique by insisting that, when x is a dyadic rational, we choose the expansion terminating in zeros. Define

$$X(x) = x, \quad X_n(x) = x_n, \quad \mathcal{F}_n = \sigma(X_1, \dots, X_n).$$

- (a) Show that $(X_n : n \in \mathbb{N})$ is a sequence of independent random variables.
- (b) Show that $\mathcal{F}_n$ is generated by a finite measurable partition of Ω .
- (c) Show that

$$\mathbb{E}(X|\mathcal{F}_n) = \sum_{k=1}^n 2^{-k} X_k + 2^{-n-1} \quad \text{almost surely.}$$

2. Let $(X_n)_{n \geq 1}$ be a sequence of independent identically distributed random variables, having finite mean μ and variance σ^2 . Set

$$S_0 = 0, \quad S_n = X_1 + \cdots + X_n, \quad M_n = S_n - \mu n, \quad Q_n = M_n^2 - \sigma^2 n.$$

Fix $\theta \in \mathbb{R}$ and assume that

$$\psi(\theta) = \log \mathbb{E}(e^{\theta X_1}) < \infty.$$

Set

$$Z_n = e^{\theta S_n - \psi(\theta)n}.$$

Show that the processes $(M_n)_{n \geq 0}$, $(Q_n)_{n \geq 0}$ and $(Z_n)_{n \geq 0}$ are all martingales, with respect to a common filtration, to be specified.

3. (a) Let X be a square integrable random variable and let $\mathcal{G}$ be a sub- σ -algebra of $\mathcal{F}$. Show that $\mathbb{E}(X|\mathcal{G})^2 \leq \mathbb{E}(X^2|\mathcal{G})$ almost surely.
- (b) Let $(M_n)_{n \geq 0}$ be a square integrable martingale. Show that $(M_n^2)_{n \geq 0}$ is a submartingale.
- (c) Let $(X_n)_{n \geq 0}$ be a submartingale starting from 0. Show that $(X_n)_{n \geq 0}$ can be written uniquely (in a suitable sense) in the form $X_n = M_n + A_n$, where $(M_n)_{n \geq 0}$ is a martingale starting from 0 and $(A_n)_{n \geq 0}$ is a non-decreasing previsible process starting from 0.
- (d) Identify $(M_n)_{n \geq 0}$ and $(A_n)_{n \geq 0}$ in the case where $X_n = S_n^2$ and $(S_n)_{n \geq 0}$ is a simple symmetric random walk on the integers.

4. Let $(X_n)_{n \geq 0}$ be a simple random walk on the integers starting from 0, with

$$p = \mathbb{P}(X_1 = 1) = 1 - \mathbb{P}(X_1 = -1) \in (0, 1).$$

Fix integers $a, b \geq 1$ and set

$$T = \min\{n \geq 0 : X_n \in \{-a, b\}\}.$$

You may assume throughout that $T < \infty$ almost surely, as you know from Markov Chains. Set

$$M_n = X_n - (2p - 1)n, \quad Q_n = M_n^2 - 4p(1 - p)n, \quad Z_n = \lambda^{X_n}, \quad \lambda = \frac{1 - p}{p}.$$

(a) Show that the processes $(M_n)_{n \geq 0}$, $(Q_n)_{n \geq 0}$ and $(Z_n)_{n \geq 0}$ are all martingales.

(b) Consider the case $p = 1/2$. Use the optional stopping theorem to show that, for all $n \geq 0$,

$$\mathbb{E}(X_{T \wedge n}) = 0, \quad \mathbb{E}(X_{T \wedge n}^2) = \mathbb{E}(T \wedge n).$$

Hence find $\mathbb{P}(X_T = b)$ and $\mathbb{E}(T)$, being careful to justify any limit arguments you use.

(c) Consider the case $p \neq 1/2$. Use the optional stopping theorem to show that

$$\mathbb{P}(X_T = b) = \frac{1 - \lambda^{-a}}{\lambda^b - \lambda^{-a}}$$

and

$$\mathbb{E}(T) = \frac{(1 - \lambda^{-a})b - (\lambda^b - 1)a}{(2p - 1)(\lambda^b - \lambda^{-a})}.$$

5. Let $(X_n)_{n \geq 0}$ be a Markov chain with finite state-space E and transition matrix P . Set $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$. Given a function f on E , define a function Pf on E by

$$Pf(x) = \sum_{y \in E} p_{xy} f(y).$$

(a) Show that $\mathbb{E}(f(X_{n+1}) | \mathcal{F}_n) = Pf(X_n)$ almost surely.

(b) Show that the following process is a martingale

$$M_n = f(X_n) - f(X_0) - \sum_{k=0}^{n-1} (P - I)f(X_k).$$

6. Consider the single-period binomial model $S_0^0 = S_0 = 1$ and $S_1^0 = 1 + r > 0$, with

$$S_1 = \begin{cases} 1 + a, & \text{with probability } 1 - p, \\ 1 + b, & \text{with probability } p \end{cases}$$

for some $a < b$ and some $p \in (0, 1)$. Show that if $r \leq a$ or $r \geq b$ then this model has an arbitrage.

7. Consider a single-period model with two assets: a riskless bond for which $S_0^0 = 1$ and $S_1^0 = 1+r > 0$, and a risky asset for which $S_0 = 1$ and S_1 takes three values $a < b < c$ with positive probability. Assume that $a < 1+r < c$.

- (a) Determine all equivalent martingale measures for this model.
- (b) Find all functions f such that the contingent claim $C = f(S_1)$ has a replicating portfolio θ .
- (c) Show directly that the initial value of the replicating portfolio in (b) is given by $\mathbb{E}(C)/(1+r)$ for all equivalent martingale measures $\mathbb{P}$.

8. Consider a single-period model with two assets: a riskless bond for which $S_0^0 = 1$ and $S_1^0 = 4/3$, and a risky asset for which $S_0 = 5$ and S_1 takes the values 4 and 7 with positive probability. Determine the no-arbitrage initial price for a time-1 call option with strike 6, and find a replicating portfolio for this option. Determine also the no-arbitrage initial price for a time-1 put option with the same strike.

Suppose we now change the model by adding 6 as a value taken by S_1 with positive probability. Show directly that the call option cannot now be replicated in the market. Suppose that the call is offered at an initial price P_0 . Show that this presents an arbitrage opportunity if and only if $P_0 \notin (1/2, 2/3)$.

9. Consider the following single-period model: there is a riskless bond with interest rate $r = 1/4$ and there are two risky assets S^1 and S^2 such that $S_0 = (S_0^1, S_0^2) = (4, 4)$ and $S_1 = (S_1^1, S_1^2)$ takes each of the values $(3, 9)$, $(6, 8)$ and $(6, 4)$ with positive probability. Find all equivalent martingale measures and all arbitrage portfolios for the model.

10. Consider a single-period model with riskless bond $S_0^0 = S_1^0 = 1$ and risky asset given by $S_0 = 1$ and $S_1 = e^{\sigma Z + \mu}$ where $Z \sim N(0, 1)$ under $\mathbb{P}$ for some $\mu \in \mathbb{R}$ and $\sigma > 0$. Fix $\alpha > 0$ and define $\tilde{\mathbb{P}}$ by $d\tilde{\mathbb{P}}/d\mathbb{P} \propto e^{\alpha Z}$. Show that $\tilde{\mathbb{P}}$ is an equivalent martingale measure for the model for some α , to be determined.