KUWAIT FOUNDATION LECTURE 9 - October 2, 2000

SPECIAL VALUES OF ZETA FUNCTIONS

Stephen Lichtenbaum

(Brown University)

It has been known for a long time that special values of the zeta function $\zeta_F(s)$ of a number field F provide arithmetical information about its ring of integers $\mathcal{O}_F$. The classical example is obtained by combining the formula of Dirichlet giving the residue of $\zeta_F(s)$ at $s = 1$ with the functional equation for $\zeta_F(s)$ to obtain that ζ_F has a zero of order $r_1 + r_2 - 1$ at $s = 0$ (where $F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$) and $\lim_{s\to 0} \zeta_F(s)s^{-(r_1+r_2-1)} = -hR/w$, where h is the class number of F, R is the regulator of F, and w the number of roots of unity of F.

About thirty years ago I conjectured, somewhat rashly, that similar formulas held for all negative integers. More precisely, one should have

$$\zeta_F^*(i) = \lim_{s\to -i} \zeta_F(s)(s+i)^{-m(i)} = \pm \frac{\#K_{2i}(\mathcal{O}_F)}{\#K_{2i+1}(\mathcal{O}_F)_{\text{ tor}}} \cdot R_i,$$

Where $m(i)$ is the order of the zero of $\zeta_F(s)$ at $s = -i$ (known to be the rank of $K_{2i+1}(\mathcal{O}_F)$ by a theorem of Borel), the $K_j(\mathcal{O}_F)$ are the algebraic K-groups of $\mathcal{O}_F$, and R_i is a "higher K-theoretic regulator".

It is now clear that the conjectures should really have concerned motivic cohomology groups rather than K-groups, although it is still possible that, up to 2-torsion, the above K-groups are isomorphic to the corresponding motivic cohomology groups, so the original conjecture may still be true up to powers of 2.

In their modified motivic cohomology form, a proof of these conjectures (up to 2-torsion) for abelian number fields F has been announced by Kolster and his collaborators, developing ideas of Mazur, Wiles, Soulé, Beilinson, Deligne, Huber and Kings.

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