Hilbert's 14^{th} problem and Verlinde type formulas for rings of invariant polynomials

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Hilbert's 14^{th} problem asks if the ring $\mathbb{C}[m]^S$ of polynomials which are invariant by a set S of $m \times m$ regular matrices is finitely generated or not as ring, where $\mathbb{C}[m]$ denotes the polynomial ring of m variables. This problem is reduced to the crucial case where S consists of unipotent upper triangular matrices by virtue of Hilbert's theorem. The answer to this crucial case is affirmative when $\#S = 1$ and negative when $\#S \geqslant 3$. But the following seems still open:

PROBLEM Is the ring $\mathbb{C}[m]^{A,B}$ of polynomials which are invariant by a pair (A, B) of unipotent upper triangular matrices with $[A, B] = 0$ finitely generated?

In the following two cases the answer is affirmative, and the ring $\mathbb{C}[m]^{A,B}$ has a dimension formula similar to the Verlinde formula of conformal blocks. In both cases, $\mathbb{C}[m]^{A,B}$ is $\mathbb{Z}$-graded by *level*, and the dimension of the level l part is given as follows:

$$(1)\ \text{For } m = 2n,\quad \text{if } A = \left(\begin{array}{c|c} I_n & I_n \\ \hline 0 & I_n \end{array}\right) \text{ and } B = \left(\begin{array}{c|ccc} I_n & b_1 & & \\ & & \ddots & \\ & & & b_n \\ \hline 0 & & I_n & \end{array}\right)$$

with $b_1, \dots, b_n$ all distinct, then

$$\dim\ \mathbb{C}[2n]_l^{A,B} = \frac{1}{(l+2)2^{n-2}} \sum_{j=1}^{l+1} \frac{\cos^2 \frac{(2j+1)\pi}{2l+4}}{\sin^{2n-2} \frac{2j\ \pi}{2l+4}}$$

$$(2)\ \text{For } m = 2n+1,\ \text{if } A = \left(\begin{array}{cc|c} I_n & I_n & \\ 0 & I_n & \\ \hline & & 1 \end{array}\right) \text{ and } B = \left(\begin{array}{c|c|c} I_n & & I_n \\ \hline & 1 & \\ \hline 0 & & I_n \end{array}\right),\ \text{then}$$

$$\dim\ \mathbb{C}[2n]^{A,B} = \frac{1}{(2l+3)2^{2n-3}} \sum_{j=1}^{l+1} \frac{\sin^2 \frac{2j\pi}{2l+3}}{\cos^{2n-1} \frac{2j\pi}{2l+3}}.$$

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January 30, 2007

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PROBLEM Is the ring $\mathbb{C}[m]^{A,B}$ of polynomials which are invariant by a pair (A, B) of unipotent upper triangular matrices with $[A, B] = 0$ finitely generated?

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(1) $m = 2n$, $A = \left(\begin{array}{c|c} I_n & I_n \\ \hline 0 & I_n \end{array}\right)$, $B = \left(\begin{array}{c|c} I_n & \begin{matrix} b_1 & & \\ & \ddots & \\ & & b_n \end{matrix} \\ \hline 0 & I_n \end{array}\right)$ and $b_1, \dots, b_n$ are all distinct.

$$\dim \mathbb{C}[2n]^{A,B}_{l} = \frac{1}{(l+2)2^{n-2}} \sum_{j=1}^{l+1} \frac{\cos^2 \frac{(2j+1)\pi}{2l+4}}{\sin^{2n-2} \frac{2j\pi}{2l+4}}$$

(2) $m = 2n+1$, $A = \left(\begin{array}{cc|c} I_n & I_n & \\ 0 & I_n & \\ \hline & & 1 \end{array}\right)$, $B = \left(\begin{array}{c||c} I_n & I_n \\ \hline\hline 0 & I_n \end{array}\right)$

$$\dim \mathbb{C}[2n]^{A,B} = \frac{1}{(2l+3)2^{2n-3}} \sum_{j=1}^{l+1} \frac{\sin^2 \frac{2j\pi}{2l+3}}{\cos^{2n-1} \frac{2j\pi}{2l+3}}$$

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