Elliptic Curves and Main Conjectures

Christopher Skinner
(University of Michigan)

The celebrated Birch–Swinnerton-Dyer conjecture predicts precise relations between the L-function $L(E,s)$ of an elliptic curve E over $\mathbb{Q}$ and various arithmetic objects associated to E, especially its Mordell-Weil group $E(\mathbb{Q})$ and its Tate-Shafarevich group $Ш(E)$:

(1) $\mathrm{ord}_{s=1}L(E,s) = \mathrm{rank}\, E(\mathbb{Q})$

(2) $\lim_{s\to 1}(s-1)^{-\mathrm{rank}}\frac{L(E,s)}{\mathrm{period}} = \#Ш(E) \times (\text{regulator, etc.})$.

One approach to studying such problems about "special values", one that had great success in studying values of Dirichlet L-functions, is Iwasawa theory. Roughly this is a systematic analysis of the variation of the p-parts of the quantities in question, for a fixed prime p. In the case of elliptic curves, assuming E has ordinary reduction at p, the L-values $L(E,1)/\mathrm{period}$ get packaged into a p-adic L-function $\mathfrak{L}_E$, and $E(\mathbb{Q})$ and $Ш(E)$ together get replaced by a Selmer group $\mathrm{Sel}_p(\mathbb{Q}_\infty, E)$ ($\mathbb{Q}_\infty/\mathbb{Q}$ being the $\mathbb{Z}_p$-extension of $\mathbb{Q}$). The Main Conjecture of the title is that the characteristic ideal char_E of $\mathrm{Sel}_p(\mathbb{Q}_\infty, E)$ as a $\mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]]$-module is generated by $\mathfrak{L}_E$. K. Kato has made great progress towards this conjecture (discussed in the 6th Lecture), essentially showing that $\mathfrak{L}_E \in \mathrm{char}_E$.

Recently, Eric Urban and I have shown that $\mathrm{char}_E \subseteq (\mathfrak{L}_E)$ for many elliptic curves, completing the proof of the Main Conjecture in these cases. From this, one gets new results along the lines of (a) $L(E,1) = 0 \Rightarrow \mathrm{Sel}_p(\mathbb{Q}, E)$ infinite and (b) equality of the p-parts of the two sides in (2) above.

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Christopher Skinner, University of Michigan

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