ON NON-LINEAR DIFFERENTIAL GALOIS THEORY

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Let X be a complex analytic manifold; let Aut$X$ be the groupoid of germs of invertible maps $(X,a) \xrightarrow[u]{\sim} (X,b)$, with $a, b \in X$, $u(a) = b$. Roughly speaking a Lie groupoid (or "D-groupoid") is a subgroupoid of Aut$X$ defined by partial differential equations. The precise definition is in terms of jet spaces, and differential ideals in space of functions on jet spaces. (I take functions holomorphic in the coordinates of source and targets and polynomial in the derivatives.) Special examples are the algebraic subgroups of $\mathrm{GL}(n, \mathbb{C})$, and more generally the equation fixing a differential structure on X (e.g. a differential form, a tensor, etc.).

A D-groupoid has a "D-Lie-algebra" which is a subsheaf of $\mathcal{T}_X$ (vector fields on X), stable by Lie bracket, and defined by linear p.d.e.'s. But, in general, the converse is not true: given a D-Lie-algebra L, I call its 'enveloppe' the smallest D-groupoid whose Lie algebra contains L. (A non-trivial point is to prove its existence; this requires an analytic version of the theory of differential ideals of Ritt).

The most interesting examples of "enveloppes" are obtained when one takes for L the D-Lie-algebra defined by a foliation with singularities. One has easily a groupoid containing the enveloppe, i.e. the groupoid of automorphisms of the foliation. To reduce this groupoid to a smaller one is essentially equivalent to finding transversal structures to the foliation (e.g. symplectic, projective, Riemannian, etc.).

At the moment, few examples are well understood:

1. Linear equations. Then, the enveloppe of the corresponding foliation is essentially equivalent to the linear Galois group of the equation.

2. Foliations of codimension 1. Except the trivial case (enveloppe = all automorphisms of the foliation), the transversal dimension of the groupoid can only be 0,1,2,3. Case 0 (resp. 1) is related to the existence of a first integral (resp. an integrating factor). Cases 2 and 3 are related to Godbillon-Vey sequences of length 2 or 3 associated to the foliation.

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