KUWAIT FOUNDATION LECTURE 31 - May 22, 2003

4-DIMENSIONAL EINSTEIN MANIFOLDS

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One of the main themes in Riemannian geometry is the relationship between curvature and topology. The theory of 2-manifolds provides two main sources of inspiration: the Gauss-Bonnet theorem, and the uniformization theorem. From these we learn that every compact manifold M^2 (without boundary) admits a metric of constant curvature, and that, if we normalize the total 2-volume (=area) to be 1, the value of this constant curvature is determined by the topology. In this lecture, I developed the idea of an Einstein metric — that is, a metric of constant Ricci curvature — and explored the question of when a compact n-manifold admits such a metric. For $n = 4$, the answer is dependent on the diffeomorphism type of the manifold, and not just its homeomorphism type. In particular, a 4-manifold M with fixed orientation can admit an Einstein metric only if its Euler characteristic χ and signature τ satisfy $2\chi + 3\tau \geqslant \frac{2}{3}\alpha^2(M)$ and $2\chi - 3\tau \geqslant \frac{1}{3}\alpha^2(M)$, where α is a diffeomorphism invariant defined using the Seiberg-Witten equations.

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