

p -ADIC GALOIS REPRESENTATIONS

Jean-Marc Fontaine
(Université Paris XI)

There are conjectural descriptions of ℓ -adic representations of the absolute Galois group G_K of a field K “coming from algebraic geometry” if K is a finite field or if K is a number field. In the first case, it relies on the notion of weight. In the second one, this is a conjecture of Mazur and me: a representation is geometric if it is unramified away from finitely many primes and de Rham at all primes above ℓ .

Let B_{dR} be the field of p -adic periods corresponding to $\overline{\mathbb{Q}_p}/\mathbb{Q}_p$. This is a complete discrete valuation field whose residue field is the p -adic completion \mathbb{C}_p of $\overline{\mathbb{Q}_p}$ and on which $G_p = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ acts. For any p -adic representation V of G_K of dimension h , $D_{dR}(V) := (B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_p}$ is a \mathbb{Q}_p -vector space of dimension $\leq h$. One says that V is de Rham if equality holds.

Granted two recent results (Theorems A and B below), the category of de Rham representations is equivalent to a category lying in the world of linear algebra. In turns, the proofs of these use deep p -adic analysis:

- the proof of Theorem B (due to Colmez and me) uses the new notion of Banach-Colmez spaces which are p -adic Banach spaces equipped with a suitable analytic structure over \mathbb{C}_p ,
- the proof of Theorem A (due to Berger) relies on the theory of (φ, Γ) -modules (which gives a complete classification of all p -adic representations of G_p) and on a deep result on p -adic differential equations proved independently by André, Kedlaya and Mebkhout.

The field B_{dR} contains a sub- \mathbb{Q}_p -algebra B_{st} , stable under G_p , equipped with two operators φ and N , satisfying $N\varphi = p\varphi N$.

For any finite Galois extension L of \mathbb{Q}_p contained in $\overline{\mathbb{Q}_p}$, $B_{st}^{\text{Gal}(\overline{\mathbb{Q}_p}/L)} = L_0$, the maximal unramified extension of \mathbb{Q}_p contained in L and B_{st} ; $L(V) := (B_{st} \otimes_{\mathbb{Q}_p} V)^{\text{Gal}(\overline{\mathbb{Q}_p}/L)}$ is an L_0 -vector space of dimension $\leq h$. One says that V is L -semi-stable if equality holds. Any L -semi-stable representation is de Rham and, conversely, Theorem A asserts that any de Rham representation is L -semi-stable, for L big enough.

Theorem B gives an equivalence between the category of L -semi-stable p -adic representations of G_p and the category of admissible filtered $(\varphi, N, \text{Gal}(L/\mathbb{Q}_p))$ -modules. In the special case $L = \mathbb{Q}_p$ (semi-stable case), an object of this category is a finite dimensional \mathbb{Q}_p -vector space D equipped with two endomorphisms φ and N , with φ injective and $N\varphi = p\varphi N$, and a decreasing filtration $(\text{Fil}^r D)_{r \in \mathbb{Z}}$ by sub- \mathbb{Q}_p -vector spaces with $\text{Fil}^r D = 0$ for $r \gg 0$, $\text{Fil}^r D = D$ for $r \ll 0$, such that $t_H(D) = t_N(D)$ and $t_H(D') \leq t_N(D')$ for D' sub- \mathbb{Q}_p -vector space stable under φ and N . (Here $t_H(D') = \sum_{r \in \mathbb{Z}} r \cdot \dim_{\mathbb{Q}_p} \text{Fil}^r D' / \text{Fil}^{r+1} D'$ and $t_N(D') = v_p(\det \varphi)$.)

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"p-adic Galois representations"

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Theorem B gives an ~~explicit description~~ equivalence between the category of L-semi-stable p-adic representations of G_p and the category of admissible filtered (φ, N) -modules. In the special case $L = \mathbb{Q}_p$ (semi-stable case), an object of this category is a finite dim \mathbb{Q}_p -vector space D equipped with two endomorphisms φ and N , with φ injective and $N\varphi = p\varphi N$, and a decreasing filtration $(\text{Fil}^i D)_{i \in \mathbb{Z}}$ by sub- \mathbb{Q}_p -vector spaces with $\text{Fil}^i D = 0$ for $i \gg 0$, $\text{Fil}^i D = D$ for $i \ll 0$ such that

$$t_H(D) = t_N(D) \text{ and } t_H(D') \leq t_N(D') \text{ for } D' \text{ sub-}\mathbb{Q}_p\text{-vector space stable under } \varphi \text{ and } N$$

$$\text{Check } t_H(D) = \sum_{i=0}^1 i \cdot \dim_{\mathbb{Q}_p} \text{Fil}^i D / \text{Fil}^{i+1} D \text{ and } t_N(D) = v_p(\det \varphi)$$