KUWAIT FOUNDATION LECTURE 24 - April 23, 2002

INTEGRAL p-ADIC HODGE THEORY

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p-adic Hodge theory was founded by Tate and Grothendieck in the late sixties and precise and general conjectures were formulated by Fontaine in the late seventies. These conjectures were proven during the last decade, making p-adic Hodge theory a "complete working" theory. However, for the need of arithmetics, it was clear for long that this theory although a powerful tool, was sometimes not sufficient: it could happen one needed to work with integral structures and not just vector spaces (eg. A. Wiles & R. Taylor's proof of Fermat Last Theorem) i.e. one needed some "integral p-adic Hodge theory".

The aim of this lecture is to give the "state of the art"concerning integral p-adic Hodge theory. The first part is devoted to the statement of the 2 main results of p-adic Hodge theory: (1) the Colmez-Fontaine theorem describing p-adic semi-stable representations of the Galois group of a local field of char. 0 with perfect residue field of char. p in terms of (semi-)linear data called "(φ, N)-filtered modules"and (2) the Tsuji theorem saying that the p-adic étale cohomology of a proper smooth scheme over the above local field with semi-stable reduction is indeed a semi-stable representation with associated filtered module being essentially the de Rham cohomology.

The second part of the lecture is devoted to conjectures and results in integral p-adic Hodge theory. Under the assumption that the Hodge-Tate weights of the p-adic representations are small enough (essentially between 0 and p):

(1) a conjectural description of all Galois stable $\mathbb{Z}_p$-lattices in semi-stable representations is given in terms of integral structures on the filtered modules side called "strongly divisible lattices" and

(2) a conjectural geometric interpretation of these strongly divisible lattices is given in case the Galois lattice is étale cohomology with $\mathbb{Z}_p$-coefficients (mod torsion).

These two conjectures are theorems either if the local field is absolutely unramified or if the Hodge-Tate weights are between 0 and 1. This last case is a consequence of a new classification of p-divisible groups (and, further, of commutative finite flat group schemes) over the ring of integers of the local field without restriction on the ramification. This classification was one of the tools in the study of the last cases of Shimura-Taniyama-Weil by B. Conrad, F. Diamond, R. Taylor and myself, following A. Wiles' method.

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"Integral p-adic Hodge theory"

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