KUWAIT FOUNDATION LECTURE 20 - January 24, 2002

THE HAMILTON-JACOBI EQUATION: THE REVIVAL OF THE INTERPLAY BETWEEN PARTIAL DIFFERENTIAL EQUATIONS AND LAGRANGIAN DYNAMICS

Albert Fathi

(Ecole Normale Supérieure de Lyon, France)

A model problem of classical dynamics is the understanding of the trajectory of particles in $\mathbb{R}^n$ submitted to a $\mathbb{Z}^n$-periodic potential V. When the potential V is C^∞-small, KAM theory provides for some set of $p \in \mathbb{R}^n$ a $\mathbb{Z}^n$-period function U_p such that the set $\{(x, p + \mathrm{grad}_{U_p}(x)) | x \in \mathbb{R}^n\}$ is invariant under the motion. In particular, the function $f_p(x) = \langle p, x\rangle + U_p(x)$ satisfies the Hamilton-Jacobi equation:

$$H(x, \mathrm{grad} f_p(x)) = \frac{1}{2} \parallel \mathrm{grad} f_p(x) \parallel^2 + V(x) = \text{constant}.$$

For $n = 1$, in 1982, Aubry and Mather showed how to construct invariant sets for the p's not covered by KAM theory. These invariant sets were obtained from appropriate action-minimizing curves.

In 1987, Mather gave the right framework for higher dimensions and constructed the invariant sets. About the same time, Lions, Papanicolaou and Varadhan provided weak (viscosity) solutions to the HJE. The goal of the lecture was to explain how to obtain Mather invariant sets from these viscosity solutions.

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The Hamilton-Jacobi Equation: The revival of the interplay between Partial Differential Equations and Lagrangian Dynamics

A model problem of Classical dynamics is the understanding of the trajectory of particles in $\mathbb{R}^n$ submitted to a $\mathbb{Z}^n$-periodic potential V. When the potential V is C^∞-small, KAM theory provides for some set of $p \in \mathbb{R}^n$ a $\mathbb{Z}^n$-periodic function u_p such that the set $\{(x, p + \mathrm{grad}\, u_p(x)) \mid x \in \mathbb{R}^n\}$ is invariant under the motion. In particular, the function $f_p(x) = \langle p, x \rangle + u_p(x)$ satisfies the Hamilton Jacobi equation:

$$H(x, \mathrm{grad}\, f_p(x)) = \frac{1}{2} \|\mathrm{grad}\, f_p(x)\|^2 + V(x) = \text{Constant}$$

For $n=2$, in 1982, Aubry and Mather showed how to construct invariant sets for the p's not covered by KAM theory. These invariant sets were obtained from appropriate action-minimizing curves. In 1987, Mather gave the right framework for higher dimensions and constructed the invariant sets. About the same time, Lions, Papanicolaou & Varadhan provided weak (viscosity) solutions to the HJE. The goal of the lecture was to explain how to obtain Mather invariant sets from these viscosity solutions.

Albert Fathi, 24/01/02

ENS Lyon