

GALOIS REPRESENTATIONS AND DIFFERENTIAL EQUATIONS

Pierre Colmez

(Institut de Mathématiques de Jussieu)

Due to the total disconnectedness of the p -adic world, the theory of differential equations over a p -adic annulus is much more complicated than its complex counterpart where one can use analytic continuation to produce a monodromy matrix giving a good handle on what is going on. A substitute for analytic continuation is played by Frobenius endomorphisms and a conjecture of Crew says that, in the presence of a Frobenius, the situation is quite similar to the complex case. More precisely, if \mathcal{R} is the ring of analytic functions on an ‘infinitely thin’ annulus of radius 1 (\mathcal{R} is also known as the Robba ring), and \mathcal{D} is a differential module over \mathcal{R} with a Frobenius structure (a (φ, ∂) -module over \mathcal{R} for short), then Crew conjectures that \mathcal{D} has a basis of horizontal sections in $\mathcal{R}'[\log T]$ where \mathcal{R}' is a finite extension of \mathcal{R} coming from a separable extension of $\mathbb{F}_p((T))$. Crew’s conjecture has recently been proved by Y. André, by Z. Mebkhout and by K. Kedlaya independently. This provides a link between Galois modules for $\text{Gal}(\mathbb{F}_p((T))^{\text{un}}/\mathbb{F}_p((T)))$ and p -adic differential equations.

The theory of “fields of norms” of Fontaine-Wintenberger provides a link between the absolute Galois group $G_{\mathbb{Q}_p}$ of \mathbb{Q}_p and that of $\mathbb{F}_p((T))$. This has given rise to Fontaine’s theory of (φ, Γ) -modules which gives a complete “description” of p -adic representations of $G_{\mathbb{Q}_p}$. This theory, in turn allows one to associate to a p -adic representation of $G_{\mathbb{Q}_p}$ a (φ, ∂) -module over \mathcal{R} , which has proved very useful to study p -adic representations “coming from geometry”. In particular, these representations have a much nicer associated (φ, ∂) -module than random representations. This fact has allowed L. Berger to prove that a conjecture of Fontaine: “de Rham implies potentially semi-stable” was a consequence of Crew’s conjecture which means that this conjecture is now a theorem as well.

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Pierre COLMEZ

Galois representations and differential equations

Due to the total disconnectedness of the p -adic world, the theory of differential equation over a p -adic annulus is much more complicated than its complex counterpart where one can use analytic continuation to produce a monodromy matrix giving a good handle on what is going on. A substitute for analytic continuation is played by Frobenius endomorphisms and a conjecture of Crew says that, in the presence of a Frobenius, the situation is quite similar to the complex case. More precisely, if \mathcal{D} is the ring of analytic functions on an "infinitesimal thick" annulus of radius 1 (\mathcal{D} is also known as the Robba ring), then \mathcal{D} is a differential module over \mathcal{D} with a Frobenius structure (a (φ, δ) -module over \mathcal{D} for short), then Crew conjectures that \mathcal{D} has a basis of horizontal sections in $\mathcal{D}'[\log T]$ where \mathcal{D}' is a finite extension of \mathcal{D} coming from a separable extension of $\mathbb{F}_p((T))$. Crew's conjecture has recently been proved by Y. André, by Z. Mebkhout and by K. Kedlaya independently. This provides a link between Galois modules for $\text{Gal}(\mathbb{F}_p((T))^{1/p^\infty} / \mathbb{F}_p((T)))$ and p -adic differential equations.

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Pierre Colmez, 9/10/01