KUWAIT FOUNDATION LECTURE 15 - May 23, 2001

GAUSS-MANIN DETERMINANT AND TATE LINEAR ALGEBRA

Hélène Esnault
(University of Essen, Germany)

Residues of meromorphic differential forms on Riemann surfaces are easily defined via the local expansion. Cauchy integral formula on $\mathbb{P}^1$ and tori, together with triangulation in higher genus, yield the local to global formula:

$$(*) \qquad \sum_{x \in \text{Riemann surface}} \text{res}_x(\text{meromorphic form}) = 0.$$

Algebraically, one can easily transpose those definitions (Serre, Groupes algébriques et corps de classe, 1959), but it is difficult to show in characteristic $p > 0$ that the residue is well defined. Tate (1968) gave a completely new approach to these concepts, allowing a uniform proof of $(*)$ in characteristic 0 and $p > 0$ as well. His idea is that the residue is a trace of an operator on a infinite dimensional vector space, such as the Laurent series field $K((z))$ in a point, or the rational functions $k(X)$, which applied to these two cases, one obtains $(*)$.

If $f : X \to S$ is a smooth projective complex family of curves, and $\mathcal{U}$ is a local system on X, then the cohomology $H^i(X_s, \mathcal{U}|_{X_s})$ is a local system on S, and the determinant of this cohomology $\Sigma(-1)^i \det H^i$ is a character of the fundamental group of S. Using similarly Stokes' theorem and a triangulation of f, Bismut and co-authors gave a formula for this determinant.

If f is algebraic and (E, ∇) is an absolute connection on a Zariski open $U \subset X$, then the determinant of de Rham cohomology is a subtle invariant taking into account the irregularities of the connection.

With <u>A. Beilinson</u> and <u>S. Bloch</u>, we developed a notion of an epsilon connection

$$\varepsilon_x(E, \nabla) \in \text{rank 1 connections on the base}$$

for each closed point x of X/S, building up on a further development of Tate's ideas. It yields a local to global formula

$$(**) \qquad \det \text{DR cohomology}(E, \nabla) = \sum_{x \in \text{closed pts of } X} \varepsilon_x(E, \nabla)$$

One can further give an explicit shape for these local epsilon factors.

KUWAIT FOUNDATION LECTURE
23 May, 2001

Gauß-Manin determinant and Tate linear algebra.

Hélène Esnault, Universität Essen

Residues of meromorphic differential forms on Riemann surfaces are easily defined via the local expansion; Cauchy integral formula on $\mathbb{P}^1$ and tori, together with triangulation in higher genus, yield the local to global formula:

$$(*) \quad \sum_{x \in \text{Riemann surface}} \text{res}_x(\text{meromorphic form}) = 0$$

Algebraically, one can easily transpose those definitions (Serre, Groupes algébriques et corps de classe, 1959), but it is difficult to show in char $p>0$ that the residue is well defined. Tate (1968) gave a completely new approach to those concepts, allowing a uniform proof of $(*)$, in char 0 and $p>0$ as well. His idea is that the residue is a trace of an operator on an infinite dimensional vector space (such as the Laurent series field $k((t))$ at a point, or the rational functions $k(X)$) which, applied to those two cases, one obtains $(*)$.

If $f: X \to S$ is a smooth projective complex family of curves, and V is a local system on X, then the cohomology $H^i(X_s, V|_{X_s})$ is a local system on S, and the determinant of this cohomology $\sum (-1)^i \det H^i$ is a character of the fundamental group of S. Using similarly Stokes' theorem and a triangulation of f, Bismut and Lott gave a formula for this determinant.

If f is algebraic, and (E,∇) is an algebraic connection on a Zariski open $U \subset X$, then the determinant of de Rham cohomology is a subtle invariant taking into account the irregularities of the connection.

With A. Beilinson and S. Bloch, we developed a notion of an epsilon connection $\varepsilon_x(E,\nabla) \in$ rank 1 connection on the base for each closed point x of X/S, building up on Tate's ideas. It yields a further development of local to global formula

$$(**) \quad \det \text{DR cohomology}(E,\nabla) = \sum_{x \in \text{closed pt of } X} \varepsilon_x(E,\nabla)$$

One can further give an explicit shape for those local epsilon factors.