KUWAIT FOUNDATION LECTURE 12 - February 5, 2001

MOTIVIC COHOMOLOGY, ℓ-ADIC COHOMOLOGY AND THE TATE CONJECTURE

Bruno Kahn

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The past 10 years have seen fundamental advances in the definitions of motivic cohomology and triangulated categories of motives as envisioned by Beilinson, thanks to the work inter alia of Bloch, Friedlander, Geisser, Hanamura, Levine, Suslin and Voevodsky. It is now time to ask how close these new developments will lead us to a proof of much desired conjectures in arithmetic geometry: finite generation and finiteness conjectures, and the Beilinson conjectures on special values of L-functions.

This talk explains that the motivic picture does give an edge in the case of varieties over a finite field. It turns out that the most relevant conjecture is the following:

Conjecture 1 *Let X be a smooth projective variety over $\mathbb{F}_p$ and $n \geqslant 0$.*
a) (Tate) The order of the pole of the Hasse-Weil zeta function $\zeta(X,s)$ at $s=n$ is the rank of the group of cycles of codimension n over X modulo numerical equivalence.
b) (Beilinson) Rational and numerical equivalences agree on cycles of codimension n $(\otimes\mathbb{Q})$.

The main result is that Conjecture 1 is equivalent to:

Conjecture 2 *Let $\ell \neq p$. Then, for any $n > 0$, the "modified motivic ℓ-adic cycle map"*

$$\alpha^*\mathbb{Z}(n) \otimes \mathbb{Z}_\ell(0)^c \to \mathbb{Z}_\ell(n)^c$$

is a quasi-isomorphism.

Here $\mathbb{Z}(n) \in D^-((\mathrm{Sm}/\mathbb{F}_p)_{\mathrm{Zar}})$ is an object representing weight n motivic cohomology, α is the projection $(\mathrm{Sm}/\mathbb{F}_p)_{\mathrm{et}} \to (\mathrm{Sm}/\mathbb{F}_p)_{\mathrm{Zar}}$ and $\mathbb{Z}_\ell(n)^c = R\varprojlim \mu_{\ell^\nu}^{\otimes n} \in D^+((\mathrm{Sm}/\mathbb{F}_p)_{\mathrm{et}})$. The map lives in $D((\mathrm{Sm}/\mathbb{F}_p)_{\mathrm{et}})$.

In particular, Conjecture 2 is independent of ℓ. It is equivalent to a purely cohomological conjecture:

Conjecture 3 *Let $\ell \neq p$. For any $n > 0$ and any finitely generated field $K/\mathbb{F}_p$, there exists an integer $N \geqslant 2n+2$ such that $\widetilde{H}^N_{\mathrm{cont}}(\partial\widehat{\square}^q_K, \mathbb{Q}_\ell(n)) = 0$ for any $q \geqslant 0$.*

Here $\partial\widehat{\square}^q_K$ is the semi-localisation of $\partial\square^q_K = \operatorname{Spec} K[t_1,\dots,t_q]/\prod(t_i^2 - t_i)$ at the $(\varepsilon_1,\dots,\varepsilon_q)$ $(\varepsilon_i \in \{0,1\})$ and, for an $\mathbb{F}_p$-scheme X,

$$\widetilde{H}^i_{\mathrm{cont}}(X, \mathbb{Q}_\ell(n)) = \varinjlim_{X\to\mathfrak{X}} H^i_{\mathrm{cont}}(\mathfrak{X}, \mathbb{Q}_\ell(n)),$$

where $\mathfrak{X}$ runs through $\mathbb{F}_p$-schemes of finite type and H^i_{cont} is ordinary (e.g. Janssen's) continuous étale cohomology.

Geisser proved that Conjecture 1 implies the Parshin conjecture: $K_j(X) \otimes \mathbb{Q} = 0$ for $j > 0$. This in turn implies the Beilinson-Soulé vanishing conjecture (and more). Conjecture 2 also implies easily the Bass conjecture rationally ($\dim_{\mathbb{Q}} K_i(X) \otimes \mathbb{Q} < \infty$ for all i, for all $X \in \mathrm{Sm}/\mathbb{F}_p$). Finally, it can be shown that Conjecture 2 implies the Lichtenbaum conjecture on finite generation and duality of étale motivic cohomology, after localizing at ℓ, plus the ℓ-primary part of his formula for the special value of $\zeta(X,s)$ at $s=n$ (X smooth projective; there are generalisations to X arbitrary).

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This talk explains that the motivic picture does give an edge in the case of varieties over a finite field. It turns out that the most relevant conjecture is the following:

? **Conjecture 1.** Let X be a smooth projective variety over $\mathbb{F}_p$ and $n \geq 0$.
a) (Tate) The order of the pole of the Hasse-Weil zeta function $z(X,s)$ at $s=n$ is the rank of the group of cycles of codimension n over X modulo numerical equivalence.
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The main result is that conjecture 1 is equivalent to

? **Conjecture 2** Let $\ell \neq p$. Then, for any $n > 0$, the "modified motivic ℓ-adic cycle map"

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Here $\mathbb{Z}(n) \in D^-((Sm/\mathbb{F}_p)_{Zar})$ is an object representing weight n motivic cohomology, α is the projection $(Sm/\mathbb{F}_p)_{et} \to (Sm/\mathbb{F}_p)_{Zar}$ and $\mathbb{Z}_\ell(n)^c = R\varprojlim \mu_{\ell^\nu}^{\otimes n} \in D^+((Sm/\mathbb{F}_p)_{et})$. The map lives in $D((Sm/\mathbb{F}_p)_{et})$.

In particular, conjecture 2 is independent of ℓ. It is equivalent to a purely cohomological conjecture:

? **Conjecture 3** Let $\ell \neq p$. For any $n > 0$ and any finitely generated field $K/\mathbb{F}_p$, there exists an integer $N \geq 2n+2$ such that $\tilde{H}^N_{cont}(\partial\hat{\square}^q_K, \mathbb{Q}_\ell(n)) = 0$ for any $q \geq 0$.

Here $\partial\hat{\square}^q_K$ is the semi-localization of $\partial\square^q_K = \operatorname{Spec} K[t_1,\dots,t_q]/\prod(t_i^2-t_i)$ at the $(\varepsilon_1,\dots,\varepsilon_q)$ ($\varepsilon_i \in \{0,1\}$) and, for an $\mathbb{F}_p$-scheme X, $\tilde{H}^i_{cont}(X,\mathbb{Q}_\ell(n)) = \varinjlim_{X\to\mathcal{X}} H^i_{cont}(\mathcal{X},\mathbb{Q}_\ell(n))$, where $\mathcal{X}$ runs through $\mathbb{F}_p$-schemes of finite type and H^i_{cont} is ordinary (e.g. Jannsen's) continuous étale cohomology.

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Bruno Kahn, 5-2-2001