BANACH SPACES WITH SMALL SPACES OF OPERATORS

W. T. GOWERS and B. MAUREY

Abstract. *For a certain class of algebras $\mathcal{A}$ we give a method for constructing Banach spaces X such that every operator on X is close to an operator in $\mathcal{A}$. This is used to produce spaces with a small amount of structure. We present several applications.*

§1. Introduction.

This paper is a continuation of [GM], in which a space X was constructed with the property that every operator from a subspace Y to X was of the form $\lambda i+S$, where i is the inclusion map and S is strictly singular. Among the easy consequences of this fact are that X contains no unconditional basic sequence, and, more generally, that X is hereditarily indecomposable: that is, no subspace of X admits any non-trivial projection.

One could say then that X has as few operators as possible, which is why it is a counterexample to many questions about general Banach spaces. However, there are other questions which assume some structure for the space and then ask whether further structure follows. For example, if X has an unconditional basis, must it be isomorphic to some proper subspace? Various *ad hoc* techniques have been developed by the authors for finding counterexamples to some of these questions. The purpose of this paper is to be more systematic. We shall present a generalization of the main result of [GM], which, roughly speaking, states that given an algebra of maps satisfying certain conditions, one can replace the multiple of the inclusion map in the statement above by the restriction to Y of some element of the algebra. This generalization has several applications. Amongst them are constructions of a new prime Banach space, a space isomorphic to its subspaces of codimension two but not to its hyperplanes and a space isomorphic to its cube but not to its square. A related argument shows that all the operators on the space constructed in [G1], which has an unconditional basis, are of the form $D+S$, where D is diagonal and S strictly singular. Note that the diagonal operators must be continuous on such a space.

The paper will be organized as follows. In the next section we shall introduce notation and some basic lemmas. These are similar to [GM] but for technical reasons it was necessary to alter certain definitions and prove statements that were not quite the same. One result is that this paper is basically self-contained. In the third section we state and prove the main result of the paper. The applications will be given in the fourth section. The fifth and final section contains a discussion of the space mentioned above with an unconditional basis. We also ask some questions about the possibility of removing some of the restrictions needed for our main result. To understand the applications of our main theorem, it is not necessary to understand the proof of the theorem, or even the definitions of the spaces guaranteed to exist by it. The reader only interested in the applications will be ready for them after reading the beginning of Section 3 and perhaps skimming very lightly over Section 2.

§2. Notation and background.

Let c_{00} be the vector space of all (real or complex depending on the context) sequences of finite support. Let $(\mathbf{e}_n)_{n=1}^{\infty}$ be the standard basis of c_{00}. Given a vector $\mathbf{a} = \sum_{n=1}^{\infty} a_n \mathbf{e}_n$ its *support*, denoted $\mathrm{supp}(\mathbf{a})$, is the set of n such that $a_n \neq 0$. Given two subsets $E, F \subset \mathbb{N}$, we say that $E < F$ if every element of E is less than every element of F. If $x, y \in c_{00}$, we say that $x < y$ if $\mathrm{supp}(x) < \mathrm{supp}(y)$. If $x_1 < \ldots < x_n$, then we say that the vectors $x_1, \ldots, x_n$ are *successive*. This definition also applies to infinite sequences in an obvious way. An infinite sequence of successive non-zero vectors is also called a *block basis* and a subspace generated by a block basis is a *block subspace*. Given a subset $E \subset \mathbb{N}$ and a vector $\mathbf{a}$ as above, we write $E\mathbf{a}$ for the vector $\sum_{n \in E} a_n \mathbf{e}_n$. That is, E also stands for the coordinate projection associated with the set. An *interval* of integers is a set of the form $\{n, n+1, \ldots, m\}$ and the *range* of a vector x, written $\mathrm{ran}(x)$, is the smallest interval containing $\mathrm{supp}(x)$.

The following collection of functions was introduced by Schlumprecht [S] (except for the technical condition (vi) below) and will be useful here. It is the set $\mathcal{F}$ of functions $f : [1, \infty) \to [1, \infty)$ satisfying the following conditions.

(i) $f(1) = 1$ and $f(x) < x$ for every $x > 1$;

(ii) f is strictly increasing and tends to infinity;

(iii) $\lim_{x \to \infty} x^{-q} f(x) = 0$ for every $q > 0$;

(iv) the function $x/f(x)$ is concave and non-decreasing;

(v) $f(xy) \leqslant f(x)f(y)$ for every $x, y \geqslant 1$;

(vi) the right derivative of f at 1 is positive.

It is easy to check that $f(x) = \log_2(x+1)$ satisfies these conditions, as does the function

$\sqrt{f(x)}$. Note also that some of the conditions above are redundant. In particular, it follows from the other conditions that $f(x)$ and $x/f(x)$ are strictly increasing.

Now let $\mathcal{X}$ stand for the set of normed spaces $(c_{00}, \|.\|)$ such that the sequence $(\mathbf{e}_n)_{n=1}^{\infty}$ is a normalized bimonotone basis. Given $X \in \mathcal{X}$ and $f \in \mathcal{F}$, we shall say that X *satisfies a lower f-estimate* if, given any vector $x \in X$ and any sequence of intervals $E_1 < \ldots < E_n$, we have $\|x\| \geqslant f(n)^{-1} \sum_{i=1}^{n} \|E_i x\|$. Equivalently, if $x_1 < \ldots < x_n$ then $\left\|\sum_{i=1}^{n} x_i\right\| \geqslant f(n)^{-1} \sum_{i=1}^{n} \|x_i\|$.

For $X \in \mathcal{X}$, $x \in X$ and every integer $N \geqslant 1$, let

$$\|x\|_{(N)} = \sup \sum_{i=1}^{N} \|E_i x\|,$$

where the supremum is extended to all sequences $E_1, \ldots, E_N$ of successive intervals. Notice that, if $X \in \mathcal{X}$, $x \in X$, $n \in \mathbb{N}$ and E is an interval, then $\|Ex\|_{(n)} \leqslant \|x\|_{(n)}$.

For $0 < \epsilon \leqslant 1$ and $f \in \mathcal{F}$, we say that a sequence $x_1, \ldots, x_N$ of successive vectors *satisfies the RIS(ϵ) condition for the function f* if there is a sequence $(2N/f'(1))f^{-1}(N^2/\epsilon^2) < n_1 < \cdots < n_N$ of integers (where $f'(1)$ is the right derivative) such that $\|x_i\|_{(n_i)} \leqslant 1$ for each $i = 1, \ldots, N$ and

$$\epsilon\sqrt{f(n_i)} > |\operatorname{ran}(\sum_{j=1}^{i-1} x_j)|$$

for every $i = 2 \ldots, N$. Observe that when $x_1, \ldots, x_N$ satisfies the RIS(ϵ) condition for some $f \in \mathcal{F}$, then $Ex_1, \ldots, Ex_N$ also does for every interval E. When the function f is clear from the context, we shall simply say that $x_1, \ldots, x_N$ satisfies the RIS(ϵ) condition.

Given $g \in \mathcal{F}$, $M \in \mathbb{N}$ and $X \in \mathcal{X}$, an *(M, g)-form on X* is defined to be a functional x^* of norm at most one which can be written as $\sum_{j=1}^{M} x_j^*$ for a sequence $x_1^* < \ldots < x_M^*$ of successive functionals all of which have norm at most $g(M)^{-1}$. Observe that if x^* is an (M, g)-form then $|x^*(x)| \leqslant g(M)^{-1}\|x\|_{(M)}$ for any x.

Lemma 1. *Let $f, g \in \mathcal{F}$ be such that $\sqrt{f} \leqslant g$. Assume that $x_1, \ldots, x_N$ satisfies the RIS(ϵ) condition for f. If x^* is a (k, g)-form for some integer $k \geqslant 2$ then*

$$|x^*(\sum_{i=1}^{N} x_i)| \leqslant \epsilon + 1 + N/\sqrt{f(k)}.$$

In particular, $|x^*(x_1 + \cdots + x_N)| < 1 + 2\epsilon$ when $k > f^{-1}(N^2/\epsilon^2)$.

Proof. Let i be such that $n_i < k \leqslant n_{i+1}$. Then, since $\|x_j\|_{c_0} \leqslant 1$ for every $j = 1, \ldots, N$, we get

$$|(x^*, \sum_{j=1}^{i-1} x_j)| \leqslant \|x^*\|_\infty \, |\operatorname{supp}(\sum_{j=1}^{i-1} x_j)| \leqslant \epsilon\sqrt{f(n_i)}/g(k) \leqslant \epsilon,$$

$|(x^*, x_i)| \leqslant 1$ and for every $j > i$

$$|(x^*, x_j)| \leqslant \|x_j\|_{(k)}/g(k) \leqslant 1/\sqrt{f(k)}.$$

□

Lemma 2. *Let $n \in \mathbb{N}$ and let $x \in X$ be a vector such that $\|x\|_{(n)} \leqslant 1$. There exists a (non-negative) measure w on $A = \operatorname{ran} x$ such that $w(A) = 1$ and such that $w(E) \geqslant \|Ex\|$ for every interval $E \subset A$ with $\|Ex\| \geqslant n^{-1}$.*

Proof. Define $w^*(E)$ for any subinterval $E \subset A$ to be $\|Ex\|$ if $\|Ex\| \geqslant n^{-1}$, and 0 otherwise. It is enough to find a measure $w \geqslant w^*$ with $w(A) = 1$.

We consider the linear programming problem of minimizing $w(A)$ subject to the family of constraints $w(E) \geqslant w^*(E)$ for every non-empty subinterval E of A. Let $\overline{w}$ be an optimal solution for this problem and let

$$\mathcal{J} = \{E;\ \overline{w}(E) = w^*(E)\}$$

be the set of active constraints for $\overline{w}$. It is a classical fact that A belongs to the closed convex cone generated by the active constraints (identifying sets with characteristic functions). We therefore have

$$A = \sum_{E \in \mathcal{J}} c_E E\ ,$$

with $c_E \geqslant 0$. Applying the difference operator $\Delta f(x) = f(x) - f(x-1)$ to both sides of the above equation, we find that if $E \in \mathcal{J}$, $c_E > 0$ and $\max E < \max A$ then there exists $F \in \mathcal{J}$ such that $c_F > 0$ and $\min F = 1 + \max E$. It follows by induction that there exist $E_1 < \ldots < E_l$ in $\mathcal{J}$ such that $A = \bigcup_{i=1}^{l} E_i$.

Since $\|x\|_{(n)} \leqslant 1$, there are at most n intervals E_i such that $w^*(E_i) > 0$, or equivalently such that $\|E_i x\| \geqslant n^{-1}$. It follows that

$$\overline{w}(A) = \sum_{i=1}^{l} \overline{w}(E_i) = \sum_{i=1}^{l} w^*(E_i) \leqslant \|x\|_{(n)} \leqslant 1.$$

□

Lemma 3. *Let $f, g \in \mathcal{F}$, $\sqrt{f} \leqslant g$, and let $x_1, \dots, x_N$ satisfy the RIS(ϵ) condition for f. Let $x = \sum_{i=1}^N x_i$ and suppose that*

$$\|Ex\| \leqslant 1 \vee \sup\big\{|x^*(Ex)| : x^* \textit{ is a } (k,g)\textit{-form}, k \geqslant 2\big\}$$

for every interval E. Then $\|x\| \leqslant (1+2\epsilon)Ng(N)^{-1}$.

Proof. Define $G(t)$ to be $t/g(t)$ when $t \geqslant 1$ and t when $t \leqslant 1$. It is easy to check that G is concave, increasing and satisfies $G(st) \geqslant G(s)G(t)$ on the whole of $\mathbb{R}_+$.

Let n_1 be the first integer appearing in the RIS condition. We know that $Nn_1^{-1} < 1$ by the RIS condition. We have $\|x\|_{(n_1)} \leqslant N$ by the triangle inequality, thus we may find by Lemma 2 a measure w on $A = \mathrm{ran}(x)$ such that $w(A) = N$ and $\|Ex\| \leqslant w(E)$ for every interval E such that $\|Ex\| \geqslant Nn_1^{-1}$. We call $w(E)$ the *weight* of E. We shall now show that $\|Ex\| \leqslant (1+2\epsilon)G(w(E))$ for every interval E such that $\|Ex\| \geqslant Nn_1^{-1}$.

If $Nn_1^{-1} \leqslant \|Ex\| \leqslant 1$, we have

$$\|Ex\| = G(\|Ex\|) \leqslant G(w(E))$$

and so the result holds in this case. Suppose that E is a minimal interval such that $\|Ex\| \geqslant Nn_1^{-1}$ for which the inequality fails. We certainly have $\|Ex\| > 1$, and by assumption $\|Ex\| > (1+2\epsilon)G(w(E)) > (1+2\epsilon)$. We therefore have a (k,g)-form x^* such that $(1+2\epsilon)G(w(E)) < |x^*(Ex)|$. By Lemma 1 and the definition of the RIS(ϵ) condition, $k \leqslant f^{-1}(N^2/\epsilon^2) \leqslant f'(1)n_1/(2N)$.

It follows that we can find $E_1 < \dots < E_k$ with $\bigcup E_i = E$ and $|x^*(Ex)| \leqslant g(k)^{-1}\sum_{i=1}^k \|E_i x\|$, by the definition of a (k,g)-form. Let $w_i = w(E_i)$ for each i and let $w = w(E)$. Since $k \geqslant 2$ we may assume that no E_i is equal to E. For each i we have either $\|E_i x\| \leqslant Nn_1^{-1}$ or, by the minimality of E, $\|E_i x\| \leqslant (1+2\epsilon)G(w_i)$. Let A be the set of i with the first property and B the complement of A. Let s be the cardinality of A.

By Jensen's inequality, we have

$$\sum_{i\in B} \|E_i x\| \leqslant (1+2\epsilon)\sum_{i \in B} G(w_i) \leqslant (1+2\epsilon)(k-s)G(w/(k-s)) \ . \tag{$*$}$$

Therefore, setting $t = s/k$ and using the lower bound on n_1, we have

$$\begin{aligned}|x^*(Ex)| &\leqslant (1+2\epsilon)(1-s/k)G(k)G(w/(k-s)) + sNn_1^{-1} \\ &\leqslant (1+2\epsilon)\big((1-t)G(w/(1-t)) + tf'(1)/2\big) \ .\end{aligned}$$

(Observe that $s < k$, as otherwise $|x^*(Ex)| \leqslant kNn_1^{-1} < 1$). Now note that $0 < f'(1)/2 \leqslant g'(1) = G(1) - G'(1)$. Since $w \geqslant 1$ and $0 \leqslant t < 1$ it follows easily from the concavity of G that $|x^*(Ex)| \leqslant (1+2\epsilon)G(w)$, contradicting our assumption about the interval E. The result follows. $\square$

Lemma 4. *Let $X \in \mathcal{X}$. Then for every $n \in \mathbb{N}$ and $\epsilon > 0$, every block subspace of X contains a vector x of finite support such that $\|x\| = 1$ and $\|x\|_{(n)} \leqslant 1+\epsilon$. Hence, for every $N \in \mathbb{N}$, every block subspace contains a sequence $x_1, \ldots, x_N$ satisfying the RIS(ϵ) condition with $\|x_i\| \geqslant (1+\epsilon)^{-1}$.*

Proof. Without loss of generality $\epsilon \leqslant 1$. Let $m \geqslant 6n/\epsilon$ be an integer. By Lemma 3 of [GM] Y contains a vector x of norm 1 which can be written as a sum $x_1 + \ldots + x_m$ of successive vectors where every x_i has norm at most $m^{-1}(1+\epsilon/3)$. Let $E_1 < \ldots < E_n$ be any sequence of intervals whose union contains the support of x and for each $j \leqslant n$ let $A_j = \{i : \mathrm{supp}(x_i) \subset E_j\}$ and let $B_j = \{i : E_j x_i \neq 0\}$. By the triangle inequality and since the basis of X is bimonotone, $\|E_j x\| \leqslant \left\|\sum_{i \in B_j} x_i\right\| \leqslant (1+\epsilon/3)m^{-1}(|A_j|+2)$. Since $\sum_{j=1}^n |A_j| \leqslant m$ we find that $\sum_{j=1}^n \|E_j x\| \leqslant (1+\epsilon/3)(1+2n/m) \leqslant 1+\epsilon$. $\square$

§3. The main result.

We begin by defining a class of spaces, the adaptations which will interest us of the space constructed in [GM].

Given two infinite sets $A, B \subset \mathbb{N}$, define the *spread from A to B* to be the map on c_{00} defined as follows. Let the elements of A and B be written in increasing order respectively as $\{a_1, a_2, \ldots\}$ and $\{b_1, b_2, \ldots\}$. Then $\mathbf{e}_n$ maps to zero if $n \notin A$, and $\mathbf{e}_{a_k}$ maps to $\mathbf{e}_{b_k}$ for every $k \in \mathbb{N}$. Denote this map by $S_{A,B}$. Let P_A be the map $S_{A,A}$, which is just the projection on to A. Note that $S_{B,C}S_{A,B} = S_{A,C}$ and so $S_{B,A}S_{A,B} = P_A$. Note also that $S_{B,A}$ is (formally) the adjoint of $S_{A,B}$.

Given any set $\mathcal{S}$ of spreads, we shall say that it is a *proper set* if it is closed under composition (note that this applies to all compositions and not just those of the form $S_{B,C}S_{A,B}$) and taking adjoints, and if, for every $(i,j) \neq (k,l)$, there are only finitely many spreads $S \in \mathcal{S}$ for which $e_i^*(Se_j) \neq 0$ and $e_k^*(Se_l) \neq 0$. A good example of such a set is the collection of all spreads $S_{A,B}$ where $A = \{m, m+1, m+2, \ldots\}$ and $B = \{n, n+1, n+2, \ldots\}$ for some $m, n \in \mathbb{N}$. This is the proper set generated by the shift operator.

Given a Banach space X satisfying a lower f-estimate for some $f \in \mathcal{F}$, and given a subspace $Y \subset X$ generated by a block basis, we will be interested in a seminorm $|||.|||$ defined on $L(Y, X)$ as follows. Let $\mathcal{L}(Y)$ be the set of sequences $(x_n)_{n=1}^{\infty}$ of successive vectors in Y such that $\|x_n\|_{(n)} \leqslant 1$. Now let $|||T||| = \sup_{\mathbf{x} \in \mathcal{L}(Y)} \limsup_n \|Tx_n\|$. The spaces we shall consider satisfy lower f-estimates. Hence, by Lemma 4, all their subspaces contain sequences in $\mathcal{L}$ with norms bounded below (by 1/2 say). In such a space, if $|||T||| < \epsilon$, then every subspace contains a vector x such that $\|Tx\| < 2\epsilon \|x\|$. In particular, if $|||T||| = 0$, then T is strictly singular.

The next theorem is the main one of the paper. For convenience, we now fix $f \in \mathcal{F}$ for the rest of the paper to be the function $f(x) = \log_2(x+1)$, as in the statement of the theorem.

Theorem 5. *Let $\mathcal{S}$ be a proper set of spreads. There exists a Banach space $X = X(\mathcal{S})$ (satisfying a lower f-estimate where $f(x) = \log_2(x+1)$) with the following three properties.*

(i) For every $x \in X$ and every $S_{A,B} \in \mathcal{S}$, $\|S_{A,B}x\| \leqslant \|x\|$, (and therefore $\|S_{A,B}x\| = \|x\|$ if $\mathrm{supp}(x) \subset A$);

(ii) If Y is a subspace of X generated by a block basis, then every operator from Y to X is in the $|||.|||$-closure of the set of restrictions to Y of operators in the algebra $\mathcal{A}$ generated by $\mathcal{S}$. In particular, all operators on X are $|||.|||$-perturbations of operators in $\mathcal{A}$.

(iii) The seminorm $|||.|||$ satisfies the algebra inequality $|||UV||| \leqslant |||U|||\,|||V|||$.

Notice a straightforward consequence of this result. If we write $\mathcal{G}$ for the $|||.|||$-completion of $\mathcal{A}$ (after quotienting by operators with $|||.|||$ zero) then $\mathcal{G}$ is a Banach algebra. Given $T \in L(X)$, we can find by (ii) a $|||.|||$-Cauchy sequence $(T_n)_{n=1}^{\infty}$ of operators in $\mathcal{A}$ such that $|||T - T_n||| \to 0$. Let $\phi(T)$ be the limit of $(T_n)_{n=1}^{\infty}$ in $\mathcal{G}$. This map is clearly well-defined. It follows easily from (iii) that it is also a unital algebra homomorphism. The kernel of ϕ is the set of T such that $|||T||| = 0$. The restriction of ϕ to $\mathcal{A}$ is the identity (or more accurately the embedding of $\mathcal{A}$ into $\mathcal{G}$). If $\mathcal{A}$ is small, then, since the kernel of ϕ consists of small operators, $L(X)$ is also small.

The first step in the proof of the theorem is to define the space $X(\mathcal{S})$. First, we recall the definition from [GM] of the *special functionals* on a space $X \in \mathcal{X}$. Let $\mathbf{Q} \subset c_{00}$ be the set of sequences with rational coordinates and maximum at most 1 in modulus. Let $J \subset \mathbb{N}$ be a set such that, if $m < n$ and $m, n \in J$, then $\log\log\log n \geqslant 2m$. Let us write J

in increasing order as $\{j_1, j_2, \ldots\}$. We shall also need $f(j_1) > 256$, where $f(x)$ is still the function $\log_2(x+1)$. Now let $K, L \subset J$ be the sets $\{j_1, j_3, j_5, \ldots\}$ and $\{j_2, j_4, j_6, \ldots\}$.

Let σ be an injection from the collection of finite sequences of successive elements of $\mathbf{Q}$ to L. Given $X \in \mathcal{X}$ and $f \in \mathcal{F}$ such that X satisfies a lower f-estimate and given an integer $m \in \mathbb{N}$, let $A_m^*(X)$ be the set of functionals of the form $f(m)^{-1}\sum_{i=1}^m x_i^*$ such that $x_1^* < \ldots < x_m^*$ and $\|x_i^*\| \leqslant 1$ for each i. Note that these functionals have norm at most 1. If $k \in \mathbb{N}$, let Γ_k^X be the set of sequences $y_1^* < \ldots < y_k^*$ such that $y_i^* \in \mathbf{Q}$ for each i, $y_1^* \in A_{j_{2k}}^*(X)$ and $y_{i+1}^* \in A_{\sigma(y^*{}_1,\ldots,y^*{}_i)}^*(X)$ for each $1 \leqslant i \leqslant k-1$. We call these *special sequences*. Let $B_k^*(X)$ be the set of functionals of the form $f(k)^{-1/2}\sum_{j=1}^k g_j$ such that $(g_1, \ldots, g_k) \in \Gamma_k^X$. These, when $k \in K$, are the *special functionals* (on X of size k). Note that if $g \in \mathcal{F}$ and $g(k) = f(k)^{1/2}$, then a special functional of size k is also a (k, g)-form.

Now, given a proper set $\mathcal{S}$ of spreads, we define the space $X(\mathcal{S})$, inductively. It is the completion of c_{00} in the smallest norm satisfying the following equation.

$$\begin{aligned}\|x\| = \|x\|_{c_0} &\vee \sup\Big\{f(n)^{-1}\sum_{i=1}^n \|E_i x\| : 2 \leqslant n \in \mathbb{N}, E_1 < \ldots < E_n\Big\} \\ &\vee \sup\Big\{|x^*(Ex)| : k \in K,\ x^* \in B_k^*(X),\ E \subset \mathbb{N} \text{ an interval}\Big\} \\ &\vee \sup\{\|Sx\| : S \in \mathcal{S}\}\end{aligned}$$

Note that the sets $E_1 < \ldots < E_n$ above are intervals. In the case $\mathcal{S} = \{Id_{c_{00}}\}$ the fourth term drops out and the definition reduces to that of the space constructed in [GM]. The fourth term is there to force $X(\mathcal{S})$ to have property (i) claimed in the theorem. It is easy to verify that this is the case. The second term ensures that X satisfies a lower f-estimate. It is also not hard to show that $X(\mathcal{S})$ is reflexive. (A proof can be found in [GM] which works in this more general context.)

We now prove a few lemmas with the eventual aim of proving that the spaces $X(\mathcal{S})$ have the second property claimed in the main theorem. The first we quote from [GM].

Lemma 6. *Let $K_0 \subset K$. There exists a function $g \in \mathcal{F}$ such that $g \geqslant \sqrt{f}$, $g(k) = \sqrt{f(k)}$ whenever $k \in K_0$ and $g(x) = f(x)$ whenever $N \in J \setminus K_0$ and x is in the interval $[\log N, \exp N]$.* □

Lemma 7. *Let $0 < \epsilon \leqslant 1$, $0 \leqslant \delta < 1$, $M \in L$ and let n and N be integers such that $N/n \in [\log M, \exp M]$ and $f(N) \leqslant (1 + \delta) f(N/n)$. Assume that $x_1, \ldots, x_N$ satisfies the*

RIS(ϵ) condition and let $x = x_1 + \ldots + x_N$. Then $\|(f(N)/N)x\|_{(n)} \leqslant (1+\delta)(1+3\epsilon)$. In particular, if $n = 1$, we have $\|(f(N)/N)x\| \leqslant (1+3\epsilon)$.

Proof. Let g be the function given by Lemma 6 in the case $K_0 = K$. It is clear that every vector Ex such that $\|Ex\| > 1$ is normed by a (k,g)-form for some k, so the conditions of Lemma 3 are satisfied. Let $E_1 < \ldots < E_n$ be successive intervals and let w be the weight function from that proof. Then $w(\operatorname{ran} x) = N$ and, using inequality $(*)$ from that proof and noting that $N/n \in [\log M, \exp M]$, we obtain

$$\begin{aligned}\sum_{i=1}^{n} \|E_i x\| &\leqslant (1+2\epsilon)(N/g(N/n) + Nn/n_1) \\ &= (1+2\epsilon)(N/f(N/n) + Nn/n_1) \\ &\leqslant (1+2\epsilon)((1+\delta)N/f(N) + Nn/2^{N^2/\epsilon^2}) \\ &\leqslant (1+3\epsilon)(1+\delta)N/f(N)\ .\end{aligned}$$

This proves the lemma. □

The key lemma used to prove that property (ii) holds is a generalization of Lemma 22 of [GM]. Note first that a proper set $\mathcal{S}$ of spreads must be countable, and if we write it as $\{S_1, S_2, \ldots\}$ and set $\mathcal{S}_m = \{S_1, \ldots, S_m\}$ for every m, then for any $x \in X(\mathcal{S})$, $x^* \in X(\mathcal{S})^*$, we have $\lim_m \sup\{|x^*(Ux)| : U \in \mathcal{S} \setminus \mathcal{S}_m\} \leqslant \|x\|_\infty \|x^*\|_\infty$.

Lemma 8. *Let $\mathcal{S}$ be a proper set of spreads, let $X = X(\mathcal{S})$, let $Y \subset X$ be an infinite-dimensional block subspace and let T be a continuous linear operator from Y to X. Let $\mathcal{S} = \bigcup_{m=1}^\infty \mathcal{S}_m$ be a decomposition of $\mathcal{S}$ satisfying the condition just mentioned. Then for every $\epsilon > 0$ there exists m such that, for every $x \in Y$ such that $\|x\|_{(m)} \leqslant 1$ and* $\operatorname{supp}(x) > \{m\}$,

$$d(Tx, m\operatorname{conv}\{\pm Ux : U \in \mathcal{S}_m\}) \leqslant \epsilon\ .$$

Proof. It is not hard to show that T can be perturbed (in the operator norm) to an operator whose matrix (with respect to the natural bases of X and Y) has only finitely many non-zero entries in each row and column. We may therefore assume that T has this property. We may also assume that $\|T\| \leqslant 1$.

Now suppose that the result is false. Then, for some $\epsilon > 0$, we can find a sequence $(y_n)_{n=1}^\infty$ with $y_n \in Y$, $\|y_n\|_{(n)} \leqslant 1$ and $\operatorname{supp}(y_n) > \{n\}$ such that, setting

$C_n = n\,\mathrm{conv}\{\pm U y_n : U \in \mathcal{S}_n\}$, we have $d(Ty_n, C_n) > \epsilon$, and also such that if z_n is any one of y_n, Ty_n or Uy_n for some $U \in \mathcal{S}_n$ and z_{n+1} is any one of y_{n+1}, Ty_{n+1} or Vy_{n+1} for some $V \in \mathcal{S}_{n+1}$, then $z_n < z_{n+1}$.

By the Hahn-Banach theorem, for every n there is a norm-one functional y_n^* such that

$$\sup\{y_n^*(x) : x \in C_n + \epsilon B(X)\} < y_n^*(Ty_n) \ .$$

It follows that $y_n^*(Ty_n) > \epsilon$ and $\sup |y_n^*(C_n)| \leqslant 1$. Therefore $|y_n^*(Uy_n)| \leqslant n^{-1}$ for every $U \in \mathcal{S}_n$. We may also assume that y_n^* is of finite support.

Given $N \in L$ define an *N-pair* to be a pair (x, x^*) constructed as follows. Let $y_{n_1}, y_{n_2}, \ldots, y_{n_N}$ be a subsequence of $(y_n)_{n=1}^\infty$ satisfying the RIS(1) condition, which implies that $n_1 > N^2$. Let $x = N^{-1} f(N)(y_{n_1} + \ldots + y_{n_N})$ and let $x^* = f(N)^{-1}(y_{n_1}^* + \ldots + y_{n_N}^*)$, where the $y_{n_i}^*$ are as above. Lemma 7 implies that $\|x\|_{(\sqrt{N})} \leqslant 8$.

If (x, x^*) is such an N-pair, then $x^* \in A_N^*(X)$ and, by our earlier assumptions about supports,

$$x^*(Tx) = N^{-1} \sum_{i=1}^{N} y_{n_i}^*(Ty_{n_i}) > \epsilon \ .$$

Similarly, $|x^*(Ux)| \leqslant N^{-2}$ for every $U \in \mathcal{S}_N$.

Let $k \in K$ be such that $(\epsilon/24) f(k)^{1/2} > 1$. We now construct sequences $x_1, \ldots, x_k$ and $x_1^*, \ldots, x_k^*$ as follows. Let $N_1 = j_{2k}$ and let (x_1, x_1^*) be an N_1-pair. Let M_2 be such that $|x_1^*(Ux_1)| \leqslant \|x_1\|_\infty \|x_1^*\|_\infty$ if $U \in \mathcal{S} \setminus \mathcal{S}_{M_2}$. The functional x_1^* can be perturbed so that it is in $\mathbf{Q}$ and so that $\sigma(x_1^*) > \max\{M_2, f^{-1}(4)\}$, while (x_1, x_1^*) is still an N_1-pair. In general, after $x_1, \ldots, x_{i-1}$ and $x_1^*, \ldots, x_{i-1}^*$ have been constructed, let (x_i, x_i^*) be an N_i-pair such that all of x_i, Tx_i and x_i^* are supported after all of x_{i-1}, Tx_{i-1} and x_{i-1}^*, and then perturb x_i^* in such a way that, setting $N_{i+1} = \sigma(x_1^*, \ldots, x_i^*)$, we have $|x_i^*(Ux_i)| \leqslant \|x_i\|_\infty \|x_i^*\|_\infty$ whenever $U \in \mathcal{S} \setminus \mathcal{S}_{N_{i+1}}$ and we also have $f(N_{i+1}) > 2^{i+1}$ and $\sqrt{f(N_{i+1})} > 2|\operatorname{ran}(\sum_{j=1}^i x_j)|$.

Now let $x = (x_1 + \ldots + x_k)$ and let $x^* = f(k)^{-1/2}(x_1^* + \ldots + x_k^*)$. Our construction guarantees that x^* is a special functional, and therefore of norm at most 1. We therefore have

$$\|Tx\| \geqslant x^*(Tx) > \epsilon k f(k)^{-1/2} \ .$$

Our aim is now to get an upper bound for $\|x\|$ and therefore an arbitrarily large lower bound for $\|T\|$. For this purpose we use Lemma 3.

Let g be the function given by Lemma 6 in the case $K_0 = K \setminus \{k\}$. It is clear that all vectors Ex are either normed by (M, g)-forms or by spreads of special functionals of length k, or they have norm at most 1. In order to apply Lemma 3 with this g, it is therefore enough to show that $|U^* z^*(Ex)| = |z^*(UEx)| \leqslant 1$ for any special functional z^* of length k and $U \in \mathcal{S}$. Let $z^* = f(k)^{-1/2}(z_1^* + \ldots + z_k^*)$ be such a functional with $z_j^* \in A_{m_j}^*$. Suppose that $U \in \mathcal{S}_{M+1} \setminus \mathcal{S}_M$, and let j be such that $N_j \leqslant M < N_{j+1}$. Let t be the largest integer such that $m_t = N_t$. Then $z_i^* = x_i^*$ for all $i < t$. For such an i, $|z_i^*(Ux_i)| = |x_i^*(Ux_i)| < N_i^{-2}$, if $M < N_i$. If $M \geqslant N_{i+1}$, then $U \notin \mathcal{S}_{N_{i+1}}$, so $|x_i^*(Ux_i)| \leqslant \|x_i\|_\infty \|x_i^*\|_\infty \leqslant 2^{-i}$. If $N_i \leqslant M < N_{i+1}$, the only remaining case, then $i = j$ and at least we know that $|z_i^*(Ux_i)| \leqslant 1$.

If $l \neq i$ or $l = i > t$, then $z_l^*(Ux_i) = U^* z_l^*(x_i)$, and we have $U^* z_l^* \in A_{m_l}^*$ for some m_l. Moreover, because σ is injective and by definition of t, in both cases $m_l \neq N_i$. If $m_l < N_i$, then, as we remarked above, $\|x_i\|_{(\sqrt{N_i})} \leqslant 8$, so the lower bound of j_{2k} for m_1 tells us that $|U^* z_l^*(x_i)| \leqslant k^{-2}$. If $m_l > N_i$, the same conclusion follows from Lemma 1. There are at most two pairs (i, l) for which $0 \neq z_l^*(UEx_i) \neq z_l^*(Ux_i)$ and for such a pair $|z_l^*(UEx_i)| \leqslant 1$.

Putting all these facts together, we get that $|z^*(UEx)| \leqslant 1$, as desired. We also know that $(1/8)(x_1, \ldots, x_k)$ satisfies the RIS(1) condition. Hence, by Lemma 3, $\|x\| \leqslant 24kg(k)^{-1} = 24kf(k)^{-1}$. It follows that $\|T\| \geqslant (\epsilon/24)f(k)^{1/2} > 1$, a contradiction. $\square$

Lemma 9. *Let $\mathcal{S}$, X, Y, T and ϵ be as in the previous lemma, let m be as given by that lemma and let $\mathcal{A}_m = m \operatorname{conv}\{\pm\mathcal{S}_m\}$. Then there exists $U \in \mathcal{A}_m$ such that $\|T - U\| \leqslant 17\epsilon$.*

Proof. As in the last lemma, we can assume that the matrix of T has only finitely many non-zero entries in each row and column. If the statement of the lemma is false, then for every $U \in \mathcal{A}_m$ there is a sequence $\underline{x} = \underline{x}_U \in \mathcal{L}$ of vectors in Y such that $\limsup_n \|(T - U)(\underline{x})_n\| > 17\epsilon$. We will write this symbolically as $\|(T - U)\underline{x}\| > 17\epsilon$. Our first aim is to show that these $\underline{x}_U$ can be chosen continuously in U. (This statement will be made more precise later.)

Let $(\mathcal{U}_j)_{j=1}^k$ be a covering of $\mathcal{A}_m$ by open sets of diameter less than ϵ in the operator norm. For every $j = 1, \ldots, k$, let $U_j \in \mathcal{U}_j$ and let $\underline{x}_j$ be a sequence with the above property with $U = U_j$. By the condition on the diameter of $\mathcal{U}_j$, we have $\left\|(T - U)\underline{x}_j\right\| > 16\epsilon$ for every $U \in \mathcal{U}_j$. Let $(\phi_j)_{j=1}^k$ be a partition of unity on $\mathcal{A}_m$ with ϕ_j supported inside $\mathcal{U}_j$ for each j.

Let $N \in L$ be greater than k and m^2. For each $j \leqslant k$, let $x_{j,n_1}, \ldots, x_{j,n_N}$ satisfy the RIS(1) condition and let $m < x_{j,n_1}$. Let $y_j = N^{-1} f(N)(x_{j,n_1} + \ldots + x_{j,n_N})$. Let this be done in such a way that $y_1 < \ldots < y_k$ and also $(T-U)x_{j,n_1} < \ldots < (T-U)x_{j,n_N}$ for every j and every $U \in \mathcal{A}_m$. Finally, let the x_{j,n_i} be chosen so that $\|(T-U)x_{j,n_i}\| > 16\epsilon$ for every $U \in \mathcal{U}_j$.

Now let us consider the vector $y(U) = \sum_{j=1}^k \phi_j(U) y_j$. By Lemma 7 we know that, for each y_j, $\|y_j\|_{(\sqrt{N})} \leqslant 8$, from which it follows that $\|y(U)\|_{(\sqrt{N})} \leqslant 8$. We shall show that $y(U)$ is a "bad" vector for U, by showing that $\|(T-U)y(U)\| > 8\epsilon$.

To do this, let $U \in \mathcal{A}_m$ be fixed and let $J = \{j : \phi_j(U) > 0\}$. Note that $\big\|(T-U)\underline{x}_j\big\| > 16\epsilon$ for every $j \in J$. For such a j and for $i \leqslant N$ let $z^*_{j,i}$ be a norm-one functional such that $z^*_{j,i}\Big((T-U)x_{j,n_i}\Big) > 16\epsilon$. Let these functionals be chosen to be successive. Let $z^*_j = f(N)^{-1}(z^*_{j,1} + \ldots + z^*_{j,N})$ and $z^* = \sum_{j \in J} z^*_j$. Then $z^*_j(T-U)y_j > 16\epsilon$, so

$$z^*\Big((T-U)y(U)\Big) = z^*\Big(\sum_{j \in J} \phi_j(U)(T-U)y_j\Big) > 16\epsilon.$$

However, $\|z^*\| \leqslant f(kN)/f(N) \leqslant 2$, proving our claim.

The function $U \mapsto y(U)$ is clearly continuous. The vector $y(U)$ satisfies $m < y(U)$, $\|y(U)\|_{(m)} \leqslant 8$, and $\|(T-U)y(U)\| > 8\epsilon$. We now apply a fixed-point theorem.

For every $U \in \mathcal{A}_m$, let $\Gamma(U)$ be the set of $V \in \mathcal{A}_m$ such that $\|(T-V)y(U)\| \leqslant 8\epsilon$. Clearly $\Gamma(U)$ is a compact convex subset of $\mathcal{A}_m$. By the previous lemma, $\Gamma(U)$ is non-empty for every U. The continuity of $U \mapsto y(U)$ gives that Γ is upper semi-continuous, so there exists a point $U \in \mathcal{A}_m$ such that $U \in \Gamma(U)$. But this is a contradiction. $\square$

Lemma 9 shows in particular that any operator $T : Y \to X$ can be approximated arbitrarily well in the $|\!|\!|.|\!|\!|$-norm by the restriction of some operator $U \in \mathcal{A}$. We have therefore finished the proof of property (ii). The proof of (iii) is much easier, and will complete the proof of Theorem 5.

Lemma 10. *The seminorm $|\!|\!|.|\!|\!|$ satisfies the algebra inequality $|\!|\!|UV|\!|\!| \leqslant |\!|\!|U|\!|\!|\,|\!|\!|V|\!|\!|$.*

Proof. This lemma is one of the main reasons for the technical differences between this paper and [GM]. To see it, pick $\epsilon > 0$ and let $(x_n)_{n=1}^\infty \in \mathcal{L}$ be a sequence such that $\|UVx_n\| \geqslant (1+\epsilon)^{-1}\,|\!|\!|UV|\!|\!|$ for every n. After suitable perturbations and selections of subsequences we may assume that the sequences $(x_n)_{n=1}^\infty$, $(Vx_n)_{n=1}^\infty$ and $(UVx_n)_{n=1}^\infty$ are successive.

Given $m \in L$ sufficiently large, construct a vector (y_m) as follows. Choose $k \geqslant 8/\epsilon$, let $M = m^k$ and let $x_{n_1}, \ldots, x_{n_M}$ satisfy the $RIS(\epsilon)$ condition. Then let $y_m = \frac{f(M)}{M}(x_{n_1} + \ldots + x_{n_M})$. For $0 \leqslant i < m^2$ and $0 \leqslant j < m^2$, we now let

$$z_{ij} = \frac{f(M)}{M} \sum_{s=im^{k-2}+jm^{k-4}+1}^{im^{k-2}+(j+1)m^{k-4}} x_{n_s} \qquad \text{and} \qquad z_i = \sum_{j=0}^{m^2-1} z_{ij} \ .$$

By Lemma 7 we know that

$$\|z_{ij}\| \leqslant (1+3\epsilon)\frac{m^{k-4}}{f(m^{k-4})}\frac{f(M)}{M} \leqslant (1+\epsilon)(1+3\epsilon)m^{-4}$$

for each i and j. It follows from the proof of Lemma 4 that $\|z_i\|_{(m)} \leqslant m^{-2}(1+\epsilon)(1+3\epsilon)(1+2m/m^2)$ for each i. If m is large enough, we then have $\|Vz_i\| \leqslant m^{-2}(1+\epsilon)^2(1+3\epsilon)\,\|V\|$ for every i. By the proof of Lemma 4 again, we find that $\|Vy_m\|_{(m)} \leqslant (1+\epsilon)^2(1+3\epsilon)(1+2m/m^2)\,\|V\|$, and hence, for m large enough, $\|UVy_m\| \leqslant (1+\epsilon)^3(1+3\epsilon)\,\|U\|\,\|V\|$.

On the other hand, for every m

$$\|UVy_m\| = \frac{f(M)}{M}\Big\|\sum_{i=1}^M UVx_{n_i}\Big\| \geqslant \frac{1}{M}\sum_{i=1}^M \|UVx_{n_i}\| \geqslant (1+\epsilon)^{-1}\,\|UV\| \ .$$

Letting ϵ tend to zero we obtain the desired inequality. $\square$

§4. Applications.

(4.1) Let $\mathcal{S} = \{Id\}$, let $X = X(\mathcal{S})$, let Y be any block subspace of X and let i be the inclusion map from Y to X. Then given any operator T from Y to X, there exists by Theorem 5, for every $\epsilon > 0$, some λ such that $\|T - \lambda i\| < \epsilon$. Since $|\lambda| \leqslant \|T\| + \epsilon$, an easy compactness argument then shows that there exists λ such that $\|T - \lambda i\| = 0$ and thus that $T - \lambda i$ is strictly singular, which is one of the main results of [GM]. It implies easily that X contains no unconditional basic sequence, is hereditarily indecomposable and is not isomorphic to any proper subspace. (The third fact is true because every operator on X must be strictly singular or Fredholm with index zero.) In this case the algebra $\mathcal{G}$ (defined just after Theorem 5) reduces to the field of scalars $\mathbb{R}$ or $\mathbb{C}$. It follows that X^n is isomorphic to X^m if and only if $m = n$. Indeed, if $n > m$, the image under ϕ of any

$T \in L(X^n, X^m)$ is a rectangular matrix $A \in M_{m,n}$ which has non-zero kernel. It follows easily that T is singular.

(4.2) Let $\mathcal{S}$ be the proper set mentioned earlier, generated by the shift, which we denote by S. That is, $\mathcal{S}$ consists of all maps of the form $S_{A,B}$ where $A = [m, \infty)$ and $B = [n, \infty)$. We will write L for the left shift, which is (formally) the adjoint of S. Then every operator in $\mathcal{S}$ is of the form $S^m L^n$. Since $SL - I$ is of rank one, every operator in $\mathcal{A}$ is a finite-rank perturbation of an operator of the form $\sum_{n=0}^{N} \lambda_n S^n + \sum_{n=1}^{N} \mu_n L^n$, so the difference is of $|||.|||$-norm zero.

Lemma 11. *Let* $U = \sum_{n=0}^{N} \lambda_n S^n + \sum_{n=1}^{N} \mu_n L^n$. *Then* $\|U\| = |||U||| = \sum_{n=0}^{N} |\lambda_n| + \sum_{n=1}^{N} |\mu_n|$.

Proof. Note that $\sum_{n=0}^{N} |\lambda_n| + \sum_{n=1}^{N} |\mu_n|$ is the norm of U considered as an operator on ℓ_1. Clearly it is enough to prove the inequality $|||U||| \geqslant \|U\|_{\ell_1 \to \ell_1}$. For notational convenience, let $\lambda_{-n} = \mu_n$ for $1 \leqslant n \leqslant N$.

For an integer $r \in L$ consider the vector $x_r = \sum_{j=1}^{r} e_{3jN}$. Since every unit vector e_i satisfies $\|e_i\|_{(n)} = 1$ for every n, we have $\|x_r\| \leqslant r/f(r)$, by Lemma 7. On the other hand, splitting Ux_r into $3rN$ singleton pieces from $N+1$ to $(3r+1)N$ gives that $\|Ux_r\| \geqslant (r/f(3Nr)) \sum_{n=-N}^{N} |\lambda_n|$. As $r \to \infty$, $f(r)/f(3Nr) \to 1$, which shows that $\|U\|$ satisfies the required inequality. To get it for $|||U|||$, let $\epsilon > 0$ and $n \in \mathbb{N}$. Then, by Lemma 7, there exists r such that $\|(f(r)/r)x_r\|_{(n)} \leqslant 1 + \epsilon$. This is also true if x_r is shifted, so the lower bound above on $\|Ux_r\|$ gives the result. $\square$

Since all powers of S and L have norm 1, it is obviously true that every operator of the form $\sum_{n=0}^{\infty} a_n S^n + \sum_{n=1}^{\infty} a_{-n} L^n$ is continuous if $\sum_{n=-\infty}^{\infty} |a_n| < \infty$. The next result gives, up to a strictly singular perturbation, the converse of this fact.

Corollary 12. *There is an algebra homomorphism and projection* $\phi : L(X) \to L(X)$ *onto the subspace consisting of Toeplitz operators with absolutely summable coefficients. If* $T \in L(X)$ *then* $|||\phi(T) - T||| = 0$.

Proof. Recall the remark following the statement of Theorem 5. In this case, by Lemma 11, the algebra $\mathcal{G}$, the $|||.|||$-completion of $\mathcal{A}$, is the same as the completion in $L(X)$ and also the completion in the operator norm on ℓ_1. Therefore $\mathcal{G}$ can be regarded as a subalgebra of $L(X)$ consisting of Toeplitz operators with absolutely summable coefficients. If we do this, then the algebra homomorphism ϕ defined after Theorem 5 is also a projection. The equation $\|\phi(T) - T\| = 0$ follows easily from the definition of ϕ. $\square$

Recall that a Banach space is said to be *prime* if it is isomorphic to every infinite-dimensional complemented subspace of itself. The only known examples before this paper were c_0 and ℓ_p $(1 \leqslant p \leqslant \infty)$. These were shown to be prime by Pełczyński [P], apart from ℓ_∞ which is due to Lindenstrauss [L]. The space X is prime by virtue of having no non-trivial complemented subspaces and being isomorphic to its subspaces of finite codimension.

Theorem 13. *The space X is prime.*

Proof. Let $P : X \to X$ be a projection. By the previous corollary the operator $\phi(P)$ is a convolution by some absolutely summable sequence $(a_n)_{n\in\mathbb{Z}}$. Moreover, $\phi(P)^2 = \phi(P)$. But the Fourier transform of the sequence $(a_n)_{n\in\mathbb{Z}}$ is a continuous function on the circle squaring to itself. Hence it is constantly zero or one. It follows that a_0 is zero or one and all the other a_n are zero. That is, $\phi(P)$ is zero or the identity. Since $P - \phi(P)$ is strictly singular, it follows that P is of finite rank or corank. Thus, if PX is infinite-dimensional, then it has finite codimension. Since the shift on X is an isometry, it follows that X and PX are isomorphic, which proves the theorem. $\square$

We note here that the argument in the above proof can be generalized to show that if m and n are integers with $m > n$, then X^n does not contain a family $P_1, \ldots, P_m$ of infinite-rank projections satisfying $P_iP_j = 0$ whenever $i \neq j$. Indeed, given any projection $P \in L(X^n)$, we can regard it as an element of $M_n(L(X))$. Acting on each entry with first ϕ and then the Fourier transform, we get a function $h \in M_n(C(\mathbb{T}))$. The map taking P to h is an algebra homomorphism so h is an idempotent. Regarding h as a continuous function from $\mathbb{T}$ to $M_n(\mathbb{C})$, we have that $h(t)$ is an idempotent in $M_n(\mathbb{C})$ for every $t \in \mathbb{T}$. By the continuity of rank for idempotents, we have that if $h(t) = 0$ for some t, then h is identically zero. But then P is strictly singular and hence of finite rank. Applying this reasoning to the family $P_1, \ldots, P_m$ above, we obtain $h_1, \ldots, h_m$ such that, for every $t \in \mathbb{T}$, $h_1(t), \ldots, h_m(t)$ is a set of non-zero idempotents in $M_n(\mathbb{C})$ with $h_i(t)h_j(t) = 0$ when $i \neq j$. But this is impossible if $m > n$. It follows that X^n and X^m are isomorphic if and only if

$n = m$.

Another simple consequence of Corollary 12 is that, up to strictly singular perturbations, any two operators on X commute. Indeed, if V and W are two operators, then $\phi(V)$ and $\phi(W)$ commute, so $\phi(VW - WV) = 0$, from which it follows that $\|VW - WV\| = 0$.

For the rest of this section, we assume that X has complex scalars. Let $\psi : L(X) \to C(\mathbb{T})$ be the composition of ϕ with the Fourier transform. Then ψ is also a continuous algebra homomorphism. Given an operator T, let K_T be the compact set of $\mu \in \mathbb{C}$ such that μ is infinitely singular for T. (Recall that this means that for every $\epsilon > 0$ there is an infinite-dimensional subspace $Y \subset X$ such that $\|Ty - \mu y\| \leqslant \epsilon \|y\|$ for every $y \in Y$.) Since $T - \phi(T)$ is strictly singular, $K_{\phi(T)} = K_T$.

Lemma 14. *The function $\psi(T)$ takes the value zero at some $\exp(i\theta)$ if and only if 0 is infinitely singular for T.*

Proof. If $\psi(T)$ takes the value zero at $\exp i\theta$, we can construct an approximate eigenvector for $\phi(T)$ with eigenvalue zero as follows. Suppose that $\phi(T)$ is convolution by the sequence $(a_n)_{n\in\mathbb{Z}}$, and let $\epsilon > 0$. We know that $\sum_{n\in\mathbb{Z}} a_n \exp(in\theta) = 0$. Let $N \in L$ and let x_N be the vector $(f(N^2)/N^2)\sum_{n=N^2}^{2N^2} \exp(in\theta)e_n$. By Lemma 7 we have $\|x_N\| = 1$. Let U be convolution by the sequence $(a_n)_{n=-N}^{N}$. If N is large enough, then $\|U - \phi(T)\| \leqslant \epsilon/2$, since $(a_n)_{n\in\mathbb{Z}}$ is absolutely summable. Moreover, all but at most $4N$ of the possible $N^2 + 2N$ non-zero coordinates of Ux_N are equal to $(f(N^2)/N^2)\sum_{n=-N}^{N} a_n \exp(in\theta)$. Taking N sufficiently large, we can therefore make $\|\phi(T) - U\|$ and $\|Ux_N\|$ as small as we like. Therefore zero is infinitely singular for $\phi(T)$. Since $\|T - \phi(T)\| = 0$, the same is true for T.

Conversely, if $\psi(T)$ never takes the value zero, then it can be inverted in $C(\mathbb{T})$. A classical result states that the Fourier transform of this inverse will also be in $\ell_1(\mathbb{Z})$, so in particular $\phi(T)$ has an inverse U which is continuous when considered as an operator on X and satisfies $U = \phi(U)$. Therefore $\phi(UT - I) = 0$, so $UT - I$ is strictly singular and zero is not infinitely singular for T. □

Corollary 15. *K_T is the image under $\psi(T)$ of the unit circle $\mathbb{T}$.*

Proof. This follows from Lemma 14 applied to the operator $T - \lambda$. □

Theorem 16. *A subspace Y of X is isomorphic to X if and only if it has finite codimension.*

Proof. Let $T : X \to Y$ be an isomorphism. Then 0 is not infinitely singular for T, so, as in the proof of Lemma 14, we can find U such that TU, UT and I are the same, up to a strictly singular perturbation. Since $TU - I$ is strictly singular, TU is Fredholm with index zero. In particular $\operatorname{codim} Y = \operatorname{codim} TX \leqslant \operatorname{codim} TUX < \infty$. As we have already mentioned, the if part follows from the existence of the isometric shift. $\square$

(4.3) Let $\mathcal{S}$ be the proper set generated by the double shift S^2. That is, $\mathcal{S}$ is as in the previous example but m and n are required to be even. We adapt a result about Fredholm operators to show that every such operator on $X(\mathcal{S})$ has even index. Suppose that this is not true and let T be an operator of odd index. By Theorem 5, and by the fact that every operator in $\mathcal{S}$ differs by a finite-rank operator from some even shift, we can find, for any $\epsilon > 0$, some linear combination U of even shifts such that $\|\!|T - U\|\!| < \epsilon$. We obtain a contradiction by showing that no such U can have odd index and that there is an $\|\!|.\|\!|$-neighbourhood of T inside which all operators do have odd index.

Lemma 17. *Let U be a Fredholm isometry on a Banach space X with a left inverse V, and let $T : X \to X$ be a Fredholm operator which can be written in the form $P(U)+Q(V)$ for polynomials P and Q. Then the index of T is a multiple of the index of U.*

Proof. Given any operator W on X, define F_W as in [GM] to be the set of $\lambda \in \mathbb{C}$ such that $W - \lambda$ is an isomorphism on some finite-codimensional subspace. (It is not hard to show that F_W is the complement of K_W defined in the last section.) Then F_W is open and $\dim\ker(W - \lambda) < \infty$ whenever $\lambda \in F_W$. Hence, the operator $W - \lambda$ is quasi-Fredholm, and the generalized index is constant on connected components of F_W.

In the case of the operator U, it is clear that F_U contains the open unit disc and the set of all λ such that $|\lambda| > 1$. Hence, either F_U is connected and the index of $U - \lambda$ is constantly zero on F_U, or $\mathbb{C} \setminus F_U = \mathbb{T}$. In the second case, the index of $U - \lambda$ is zero if $|\lambda| > 1$ and $\operatorname{ind} U$ if $|\lambda| < 1$. In either case, the only possible values are 0 and $\operatorname{ind}(U)$.

Now suppose that T is as in the statement of the lemma. For sufficiently large N, TU^N can be written $R(U)$ for some polynomial R and is still Fredholm. Writing $R(U) = c\prod_i(U - \lambda_i)$, we must have $\lambda_i \in F_U$ for TU^N to be Fredholm, so $\operatorname{ind}(U - \lambda_i)$ is either 0 or ind(U). It follows that the index of $R(U)$, and hence that of T, is a multiple of the index of U as stated. $\square$

We now use the following lemma, which is an easy variant of a standard lemma which can be found, for example, as [LT Prop. 2.c.9].

Lemma 18. *Let X and Y be any Banach spaces and let $T : X \to Y$ be a Fredholm operator. There exists $\epsilon > 0$ such that if S is any operator such that every subspace of X contains some x for which $\|Sx\| < \epsilon \|x\|$, then $T + S$ is Fredholm of the same index as T.*

Proof. Pick X_1 such that $X = X_1 \oplus \ker T$, so $T_1 = T|_{X_1}$ is an isomorphism and let $\epsilon < (1/2) \left\|T_1^{-1}\right\|^{-1}$. If $T + S$ has infinite-dimensional kernel, then so does $T_1 + S_1$ (where $S_1 = S|_{X_1}$), so there is an infinite-dimensional $Z \subset X_1$ on which $T_1 + S_1 = 0$. This contradicts our choice of ϵ. If $(T + S)$ does not have closed range, then it is standard that there exists an infinite-dimensional subspace $Z \subset X_1$ on which $\|(T + S)|_Z\| < \epsilon$. This again contradicts our choice of ϵ.

Hence ind$(T + tS)$ is defined for all $t \in [0, 1]$. This function is known to be continuous and therefore ind$(T) =$ ind$(T + S)$. $\square$

Putting these facts together, we find that no continuous operator on $X(\mathcal{S})$ can be Fredholm with odd index. We therefore have the following result.

Theorem 19. *The space $X(\mathcal{S})$ is isomorphic to its subspaces of even codimension while not being isomorphic to those of odd codimension. In particular, it is isomorphic to its subspaces of codimension two but not to its hyperplanes.*

Remarks: A slight modification of the above approach is also possible. As in the last section, one can obtain a continuous algebra homomorphism from $L(X)$ to the Toeplitz operators on X corresponding to sequences in $\ell_1(\mathbb{Z})$. The proof of Theorem 19 is then as above except that instead of Lemma 18 one can use the lemma of which it is a variant, which states that a sufficiently small perturbation in the operator norm does not change the index of a Fredholm operator. The proof of Theorem 13 gives for this space also that every complemented subspace has finite dimension or codimension. Combining this observation with Theorem 19, we see that the space has exactly two infinite-dimensional complemented subspaces, up to isomorphism. It is true for this space as well that it is isomorphic to no subspace of infinite codimension.

Note that the methods of this section generalize easily to proper sets generated by larger powers of the shift. For example, there is a space X such that two finite-codimensional subspaces are isomorphic if and only if their codimensions are equal mod 7.

(4.4) This application is more complicated than the previous ones. The aim is to construct a space X which is isomorphic to $X \oplus X \oplus X$ but not to $X \oplus X$. There is a very natural choice of $\mathcal{S}$ in this case. For $i = 0, 1, 2$ let A_i be the set of positive integers equal to $i + 1 \pmod 3$, let S_i' be the spread from $\mathbb{N}$ to A_i and let $\mathcal{S}'$ be the semigroup generated by S_0', S_1' and S_2' and their adjoints. We shall show later that this is a proper set. The space $X(\mathcal{S}')$ is easily seen to be isomorphic to its cube, and this isomorphism is achieved in a "minimal" way. (The primes in this paragraph are to avoid confusion later.)

We shall indeed consider the space $X(\mathcal{S}')$ defined above. However, we define it slightly less directly, which helps with the proof later that it is not isomorphic to its square. The algebra $\mathcal{A}'$ arising from the above definition is, if completed in the ℓ_2-norm, isometric to the Cuntz algebra $\mathcal{O}_3$, which was analysed using K-theory by Cuntz in [C]. Our proof is inspired by his paper, although K-theory is not mentioned explicitly. We shall sketch a more directly K-theoretic approach at the end of the section.

Let $\mathcal{T}$ be the ternary tree $\bigcup_{n=0}^{\infty}\{0, 1, 2\}^n$. Let Y_{00} be the vector space of finitely supported scalar sequences indexed by $\mathcal{T}$ (including the empty sequence). Denote the canonical basis for Y_{00} by $(e_t)_{t\in\mathcal{T}}$, write e for $e_\emptyset$, and denote the length of a word $t \in \mathcal{T}$ by $|t|$. If $s, t \in \mathcal{T}$, let (s, t) stand for the concatenation of s and t. We shall now describe some operators on Y_{00}.

Let S_i and T_i, for $i = 0, 1, 2$ be defined by their action on the basis as follows:

$$S_i e_t = e_{(t,i)},\ T_i e_t = e_{(i,t)}.$$

Thus S_i can be thought of as the map taking each vertex of $\mathcal{T}$ to the i^{th} vertex immediately below it, while T_i takes the whole tree on to the i^{th} branch. The adjoints S_i^* and T_i^* act in the following way: $S_i^* e_t = e_s$ if t is of the form $t = (s, i)$, and $S_i^* e_t = 0$ otherwise, while $T_i^* e_t = e_s$ if $t = (i, s)$, and $T_i^* e_t = 0$ otherwise. The following facts are easy to check: $S_i T_j = T_j S_i$, $S_i^* S_j = T_i^* T_j = \delta_{i,j} I$; $S_i S_i^*$ and $T_i T_i^*$ are projections; if P denotes the natural rank one projection on the line $\mathbb{C}e$, then $\sum_{i=0}^{2} S_i S_i^* = \sum_{i=0}^{2} T_i T_i^* = I - P$. Let $\mathcal{S}$ and $\mathcal{A}$ be respectively the proper set generated by S_0, S_1 and S_2, and the algebra generated by this proper set. (Strictly speaking, $\mathcal{S}$ is not a proper set, but it is easy to embed $\mathcal{T}$ into $\mathbb{N}$ so that the maps S_0, S_1 and S_2 become spreads as defined earlier. One can use the above relations to check the technical condition, but we do not need this. Note that $\mathcal{S}$ is the semigroup generated by the S_i and the S_i^*, that it contains I and that $\mathcal{A}$ contains P, as we have just shown.)

In order to obtain the space X, consider the subset $\mathcal{T}'$ of $\mathcal{T}$ consisting of all words $t \in \mathcal{T}$ that do not start with 0 (including the empty sequence). We modify the definition of S_0 slightly, by letting $S_0'e$ equal e instead of e_0. Operators S_1' and S_2' are defined exactly as S_1 and S_2 were. We still have that the $S_i'S_i'^*$ are projections and that $S_i'^*S_j' = \delta_{i,j}I$, but this time $\sum_{i=0}^2 S_i'S_i'^* = I$. To each $s = (i_1, \ldots, i_n) \in \mathcal{T}'$ we can associate the integer $n_s = 3^{n-1}i_1 + \cdots + 3i_{n-1} + i_n + 1$, (with $n_\emptyset = 1$), and this defines a bijection between $\mathcal{T}'$ and $\mathbb{N}$. The operators S_0', S_1' and S_2' then coincide with the spreads on c_{00} defined earlier (in fact, $S_i'e_n = e_{3n-2+i}$), so we can define $\mathcal{S}'$ to be the proper set they generate and obtain the space $X(\mathcal{S}')$. Let $\mathcal{A}'$ be the algebra generated by $\mathcal{S}'$. We now check that $\mathcal{S}'$ is a proper set by verifying the technical condition from the definition. Using the relation $S_i'^*S_j' = \delta_{i,j}I$ one finds that every element of $\mathcal{S}'$ can be written in the form UV where $U = S_{i_1}' \ldots S_{i_k}'$ and $V = {S_{j_1}'}^* \ldots {S_{j_l}'}^*$. Fix integers $m < n$. If l is larger than $\log_3(n-m)$ then at least one of Ve_m and Ve_n is zero. Moreover, if $UVe_m = e_r$ and $UVe_n = e_s$, then $U^*e_r = Ve_m$, $U^*e_s = Ve_n$ and for the same reason one of them is zero if $|s-r| < 3^k$. The technical condition follows easily.

For $t \in \mathcal{T}$, we define S_t inductively by $S_{(t,i)} = S_iS_t$. (We also let $I = S_\emptyset$.) Thus S_t takes a vertex of the tree and moves it down the path from that vertex corresponding to the word t, or in other words, $S_te_u = e_{(u,t)}$. Let S_t^* be the adjoint of S_t. Then every $U \in \mathcal{A}$ has a decomposition

$$U = \sum_{l=1}^N a_l S_{\alpha_l} S_{\beta_l}^* \ ,$$

where α_l and β_l are words in $\mathcal{T}$. Define $c(U)$ to be the smallest value of $\max_l |\beta_l|$ over all such representations of U. If $|t| > c(U)$, U is decomposed as above with $\max_l |\beta_l| < |t|$ and G_t is the set of l such that $t = (\gamma_l, \beta_l)$ for some γ_l, then

$$Ue_t = \sum_{l \in G_t} a_l e_{(\gamma_l, \alpha_l)}.$$

(Observe that $|\gamma_l| > 0$ since $|t| > c(U)$.)

We make the obvious modifications to the above definitions for $\mathcal{A}'$. The remarks are still valid, except that the actions of S_t and S_t' on e will be different if the word t begins with 0. The next lemma is similar to Lemma 11.

Lemma 20. *Let $U \in \mathcal{A}'$. Then for $|t| > c(U)$, we have the inequality $\|Ue_t\|_1 \leqslant \|\!|U|\!\|$.*

Proof. Let $|t| > c(U)$ and suppose that $Ue_t = \sum_{k=1}^{M} c_k e_{s_k}$, where the s_ks are distinct. Since $|t| > c(U)$, we have $Ue_{u,t} = \sum_k c_k e_{u,s_k}$ for every u. Pick a sequence $(u_j)_{j=1}^{\infty}$ lacunary enough to guarantee that the sequences $(e_{u_j,t})_{j=1}^{\infty}$ and $(Ue_{u_j,t})_{j=1}^{\infty}$ are successive. Then by the construction of X, we obtain the inequality

$$\|U(\sum_{j=1}^{N} e_{u_j,t})\| \geqslant \frac{N}{f(MN)} \sum_{k=1}^{M} |c_k|,$$

while for $N \in L$

$$\|\sum_{j=1}^{N} e_{u_j,t}\| \leqslant \frac{N}{f(N)}$$

by Lemma 7. Letting $N \to \infty$, this gives $\sum_{k=1}^{M} |c_k| \leqslant \|U\|$. To get the inequality for $\|\!|U|\!\|$, we also use Lemma 7. Given n and $\epsilon > 0$, it guarantees the existence of $N \in L$ such that $\|\sum_{j=1}^{N} e_{u_j,t}\|_{(n)} \leqslant (1+\epsilon)(N/f(N))$, which is enough. □

We now consider the algebra $\mathcal{A}$. Let Y be the completion of Y_{00} equipped with the ℓ_1 norm (or in other words let $Y = \ell_1(\mathcal{T})$) and let $\mathcal{E}$ denote the norm closure of $\mathcal{A}$ in $L(Y)$. Note that every S_i or T_i is an isometry on Y.

Lemma 21. *Every Fredholm operator in $\mathcal{E}$ has index 0. More generally, every Fredholm operator $T : Y^q \to Y^p$ given by a matrix in $M_{p,q}(\mathcal{E})$ satisfies $2\,\mathrm{ind}(T) = p - q$. In particular, no operator in $M_{p,q}(\mathcal{E})$ is Fredholm if $p - q$ is odd.*

Proof. Since the Fredholm index is stable under small perturbations, it is enough to consider operators in $\mathcal{A}$ (as operators on Y). For any such operator U we associate the operator

$$\tilde{U} = \sum_{i=0}^{2} T_i U T_i^*.$$

We claim that $\tilde{U}$ is a finite rank perturbation of U. It is enough to show that $\tilde{S}_i$ is a rank-one perturbation of S_i (and that $(U^*)\tilde{} = (\tilde{U})^*$). But

$$\tilde{S}_j = \sum_{i=0}^{2} T_i S_j T_i^* = S_j \sum_{i=0}^{2} T_i T_i^* = S_j(I - P) = S_j - S_j P\ .$$

Consider the projections $Q_i = T_i T_i^*$. Then $Q_i Q_j = 0$ for $i \neq j$ and

$$Y = \mathbb{C}e \oplus Q_0 Y \oplus Q_1 Y \oplus Q_2 Y.$$

Each $T_i U T_i^*$ represents an operator on $Q_i Y$, equivalent (in the obvious sense) to U on Y, and $\tilde{U}$ is 0 on the component $\mathbb{C}e$. It follows that $\mathrm{ind}(\tilde{U}) = 3\,\mathrm{ind}(U)$. On the other hand $\mathrm{ind}(\tilde{U}) = \mathrm{ind}(U)$ since it is a finite rank perturbation of U. It follows that $\mathrm{ind}(U) = 0$.

The proof is essentially the same for the more general statement. Given a matrix $A \in M_{p,q}(\mathcal{E})$, use the tilde operation on each entry. The resulting matrix is equivalent to three copies of A plus the zero matrix in $M_{p,q}$. This zero matrix contributes q to the dimension of the kernel and p to the codimension of the image, from which we obtain the equation

$$\mathrm{ind}(T) = \mathrm{ind}(\tilde{T}) = 3\,\mathrm{ind}(T) + q - p \qquad \square$$

Let $\mathcal{I}$ be the closed two-sided ideal in $\mathcal{E}$ generated by P. This ideal contains all rank-one operators of the form $e_s^* \otimes e_t$ with $s, t \in \mathcal{T}$. Hence, every finite rank operator on Y^n which is w^* continuous (considering Y^n as the dual of $(c_0)^n$) belongs to $M_n(\mathcal{I})$. Indeed, the matrix of such an operator consists of entries which are finite sums of the form $\sum_k y_k \otimes x_k$, with $y_k \in c_0$. We can approximate y_k and x_k by finitely supported sequences $\tilde{y}_k$ and $\tilde{x}_k$, and $\sum_k \tilde{y}_k \otimes \tilde{x}_k$ certainly belongs to $\mathcal{I}$. (In fact, $\mathcal{I}$ consists exactly of the compact w^*-continuous operators on ℓ_1.)

Lemma 22. *If $V \in M_n(\mathcal{E})$ is Fredholm then there exists $W \in M_n(\mathcal{I})$ such that $V + W$ is invertible in $M_n(\mathcal{E})$.*

Proof. By Lemma 21 the index of V is zero. Let $x_1, \ldots, x_N$ and $z_1, \ldots, z_N$ be bases for the kernel and cokernel. We can construct a w^*-continuous projection $\sum_{k=1}^N y_k \otimes x_k$ on the kernel. Then $W = \sum_{k=1}^N y_k \otimes z_k$ will do. $\square$

Let $\mathcal{O}$ denote the quotient algebra $\mathcal{E}/\mathcal{I}$.

Lemma 23. *Any lifting in $M_n(\mathcal{E})$ of an invertible element in $M_n(\mathcal{O})$ is Fredholm on Y^n.*

Proof. Let $xy = yx = 1$ in $M_n(\mathcal{O})$ and let u, v be any liftings of x and y. Then uv and vu are compact perturbations of the identity and hence Fredholm. It follows that u and v

are isomorphisms on finite codimensional subspaces and have finite dimensional cokernels. Hence, u is Fredholm. $\Box$

As an immediate consequence of the preceding two lemmas we have the following statement.

Corollary 24. *Every invertible element of $M_n(\mathcal{O})$ can be lifted to an invertible element of $M_n(\mathcal{E})$.* $\Box$

Now recall the remarks following Theorem 5. It follows easily from Lemma 20 that $|\!|\!|.|\!|\!|$ is actually a norm on $\mathcal{A}'$, so the Banach algebra $\mathcal{G}$ is the $|\!|\!|.|\!|\!|$-completion of $\mathcal{A}'$. Recall that there is a unital algebra homomorphism $\phi : L(X) \to \mathcal{G}$.

Lemma 25. *There is a norm-one algebra homomorphism θ from $\mathcal{G}$ to $\mathcal{O}$.*

Proof. Define a map $\theta_0 : \mathcal{A}' \to \mathcal{O}$ as follows. Given $U \in \mathcal{A}'$, write $U = \sum_{l=1}^{N} a_l S'_{\alpha_l} {S'_{\beta_l}}^*$ in some way, consider the corresponding sum $\sum_{l=1}^{N} a_l S_{\alpha_l} S^*_{\beta_l}$ as an element of $\mathcal{E}$ and let $\theta_0(U)$ be the image of this operator under the quotient map from $\mathcal{E}$ to $\mathcal{O}$. To see that this map is well defined, observe that for any pair of words α and β we have the equation $S'_\alpha {S'_\beta}^* = \sum_{i=0}^{2} S'_{(i,\alpha)} {S'_{(i,\beta)}}^*$. If n is sufficiently large, we can therefore write U as above in such a way that all the α_l are words of length n. Let W_n be the set of all words of length n. Then what we have said implies that U can be written as a sum $\sum_{\alpha \in W_n} S'_\alpha V^*_\alpha$, where each V^*_α is some linear combination of distinct operators of the form ${S'_\beta}^*$. It is easy to see now that $U = 0$ if and only if $V_\alpha = 0$ for every $\alpha \in W_n$, and moreover that distinct ${S'_\beta}^*$ are linearly independent. Therefore any $U \in \mathcal{A}'$ has at most one representation in the above form. In $\mathcal{A}$ we know that for any pair of words α and β the images in $\mathcal{O}$ of the operators $S_\alpha S^*_\beta$ and $\sum_{i=0}^{2} S_{(i,\alpha)} S^*_{(i,\beta)}$ are the same. It follows that θ_0 is well defined. Similarly, one can show that it is a unital algebra homomorphism.

Let P_n denote the projection on to the first n levels of the tree $\mathcal{T}$, so that $P_n \in \mathcal{I}$ for every n. If $U \in \mathcal{A}'$, then Lemma 20 implies that

$$\lim_n \|U(I - P_n)\|_{L(Y)} \leqslant |\!|\!|U|\!|\!| \ .$$

It follows that we may extend θ_0 to a norm-one homomorphism $\theta : \mathcal{G} \to \mathcal{O}$, as claimed. $\Box$

Theorem 26. *The spaces X and $X \oplus X$ are not isomorphic.*

Proof. Suppose that they were. We would then be able to find $U \in M_{2,1}(L(X))$, $U : X \to X \oplus X$ with an inverse $V \in M_{1,2}(L(X))$, $V : X \oplus X \to X$. The matrix $\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$ is an invertible element of $M_3(L(X))$ and therefore has an invertible image in $M_3(\mathcal{O})$ under $\theta \circ \phi$. By Corollary 24 we can lift this image to an invertible element $\begin{pmatrix} u & c_1 \\ c_2 & v \end{pmatrix}$ of $M_3(\mathcal{E})$, where c_1 and c_2 are compact. It follows that $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ is Fredholm, so u and v are Fredholm liftings of the images in $\mathcal{O}$ of U and V. But this contradicts the last part of Lemma 21. □

The proof of Theorem 26 generalizes in a straightforward way to give, for every $k \in \mathbb{N}$, an example of a space X such that X^n is isomorphic to X^m if and only if $m = n \pmod k$. It is likely that every Fredholm operator on the space X of this section has zero index, so that X is not isomorphic to its hyperplanes. Working with a dyadic tree may then give an example of a space X isomorphic to X^2 but not isomorphic to its hyperplanes.

To end this subsection, we explain, as promised, how K-theory can be used to prove Theorem 26. (This argument also generalizes easily to deal with the spaces mentioned in the last paragraph.) If A is a unital Banach algebra and e an idempotent in $M_n(A)$ for some $n \geqslant 1$, let $[e]$ denote the image of e in the additive group $K_0(A)$; in particular, let $[1_A]$ or simply $[1]$ denote the image in $K_0(A)$ of the unit of A.

Now suppose that Theorem 26 is false, let $U : X \to X \oplus X$ be an onto isomorphism and let $V : X \oplus X \to X$ be its inverse. In $M_2(L(X))$ we have the equations

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} U & 0 \end{pmatrix} \begin{pmatrix} V \\ 0 \end{pmatrix}; \ f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} V \\ 0 \end{pmatrix} \begin{pmatrix} U & 0 \end{pmatrix}.$$

This means that the two idempotents e and f in $M_2(L(X))$ are equivalent, and this implies by definition of the addition in K_0 that $[1] + [1] = [1]$, so $[1] = 0$ in $K_0(L(X))$. Taking the image under $\theta \circ \phi : L(X) \to \mathcal{O}$, this yields $[1_{\mathcal{O}}] = 0$ in $K_0(\mathcal{O})$. All we have to show now is that $[1_{\mathcal{O}}] \neq 0$.

For this we follow the proof given by Cuntz for Theorem 3.7 of [C]. (For the K-theory details we assume, see for example [B].) By the definition of equivalence for idempotents, $1_{\mathcal{E}} = S_i^* S_i$ and $S_i S_i^*$ are equivalent. The relation $I - P = \sum_{i=0}^{2} S_i S_i^*$ implies in $K_0(\mathcal{E})$ that

$$[1_{\mathcal{E}}] - [P] = 3[1_{\mathcal{E}}],$$

and therefore that $[P] = -2[1_{\mathcal{E}}]$. Now consider the short exact sequence

$$0 \to \mathcal{I} \xrightarrow{j} \mathcal{E} \xrightarrow{\pi} \mathcal{O} \to 0$$

and the corresponding exact sequence in K-theory

$$K_1(\mathcal{O}) \overset{\partial_1}{\to} K_0(\mathcal{I}) \overset{j_*}{\to} K_0(\mathcal{E}) \overset{\pi_*}{\to} K_0(\mathcal{O}) \overset{\partial_0}{\to} K_1(\mathcal{I}).$$

It is easy to see that $K_1(\mathcal{I}) = 0$ and $K_0(\mathcal{I}) \simeq \mathbb{Z}$ as they are for the ideal of compact operators on ℓ_2. Corollary 24 and the definition of ∂_1 immediately imply that $\partial_1 = 0$, so we get an exact sequence

$$0 \to K_0(\mathcal{I}) \overset{j_*}{\to} K_0(\mathcal{E}) \overset{\pi_*}{\to} K_0(\mathcal{O}) \to 0.$$

Now, $r = [P]$ generates $j_*(K_0(\mathcal{I})) = \ker \pi_* \simeq \mathbb{Z}$. If $0 = [1_{\mathcal{O}}] = \pi_*([1_{\mathcal{E}}])$, it follows by exactness that $[1_{\mathcal{E}}] = nr$ for some integer $n \in \mathbb{Z}$. But we know that $r = -2[1_{\mathcal{E}}]$, so $(2n+1)r = 0$, contradicting the fact that r generates a group isomorphic to $\mathbb{Z}$.

§5. Further results, remarks and questions.

(5.1) We begin by considering a space defined in [G1], which has an unconditional basis but fails to be isomorphic to any proper subspace. Here, we prove the stronger result that every operator on the space is the sum of a diagonal operator and a strictly singular one. Notice that this result is very much in the same spirit as Theorem 5. Consider the (non-proper) set $\mathcal{S}$ of all diagonal operators with ± 1 entries. Let X be the space $X(\mathcal{S})$. There is no problem about this as the conditions not satisfied by this $\mathcal{S}$ were not needed in the definition of $X(\mathcal{S})$, but rather in the proof of Theorem 5. It is easy to see that X has a 1-unconditional basis. Now let T be any operator on X with zeros down the diagonal. We shall show that T is strictly singular, which then shows that $U - \mathrm{diag}(U)$ is strictly singular for any U.

Before we state the next lemma, we remind the reader of our convention that if A is a subset of $\mathbb{N}$, then A also denotes the projection $\sum_{n=1}^{\infty} a_n e_n \mapsto \sum_{n \in A} a_n e_n$.

Lemma 27. *Let $(x_n)_{n=1}^{\infty} \in \mathcal{L}$ be a sequence of successive vectors. Let $A_n = \mathrm{supp}(x_n)$ and, for each n, let $B_n \cup C_n$ be a partition of A_n into two subsets. Then $C_n T B_n x_n \to 0$.*

Proof. If this is not true, then we can assume, after passing to a subsequence, that $\|C_n T B_n x_n\| > \epsilon$ for some fixed $\epsilon > 0$ and every n. We may also perturb T by a strictly singular amount so that the matrix of T has finitely many non-zero entries in each row and column, T still has zeros down the diagonal and the above inequality still holds. We may now pass to a further subsequence such that all x_n are disjointly supported and so are all Tx_n.

Now let $y_n = B_n x_n$ and $z_n = C_n T B_n x_n$ for every n. Then $\|y_n\|_{(n)} \leqslant 1$, $\|z_n\| \geqslant \epsilon$ and y_n and z_n are disjointly supported. Let $B = \bigcup B_n$ and $C = \bigcup C_n$ and let U be the operator CTB, so that $U(y_n) = z_n$ for every n. We now construct a special sequence in what is becoming the usual way. (See for example the proof of Lemma 8.)

Given $N \in L$, an *N-pair* is a pair (w, w^*) constructed as follows. Let $y_{n_1}, \ldots, y_{n_N}$ be a subsequence of $(y_n)_{n=1}^\infty$ satisfying the RIS(1) condition. Let $w = N^{-1} f(N)(y_{n_1} + \ldots + y_{n_N})$ and let $w^* = f(N)^{-1}(z^*_{n_1} + \ldots + z^*_{n_N})$, where each $z^*_{n_i}$ is a support functional for z_{n_i} with $\mathrm{supp}(z^*_{n_i}) \subset \mathrm{supp}(z_{n_i})$. In this case, we have $w^* \in A^*_N(X)$. By Lemma 7, we also have $\|w\|_{(\sqrt{N})} \leqslant 8$. Notice also that $w^*(Uw) = N^{-1} \sum_{i=1}^N z^*_{n_i}(z_{n_i}) > \epsilon$.

Now for any $k \in K$ we can choose a sequence $((w_i, w_i^*))_{i=1}^k$ of such pairs as follows. Let $N_1 \in L$ satisfy $\sqrt{N_1} > (2k/f'(1))2^{k^2}$ and let (w_1, w_1^*) be an N_1-pair. Perturb w_1^* slightly so that it is in $\mathbf{Q}$, it has the same support, still satisfies $w_1^*(w_1) > \epsilon$ and so that, setting $N_2 = \sigma(w_1^*)$, we have $\sqrt{f(N_2)} > 2|\operatorname{ran}(w_1)|$. Once the first $i-1$ pairs have been chosen, let $N_i = \sigma(w_1^*, \ldots, w_{i-1}^*)$, let (w_i, w_i^*) be an N_i-pair with $w_i > w_{i-1}$ and let w_i^* be perturbed so that $w_i^* \in \mathbf{Q}$, the support is the same, $w_i^*(w_i) > \epsilon$ and so that $\sqrt{f(N_{i+1})}$ will be at least $2|\operatorname{ran}(w_1 + \ldots + w_i)|$. This construction guarantees that $(w_1^*, \ldots, w_k^*)$ is a special sequence and that $\frac{1}{8}(w_1, \ldots, w_k)$ satisfies the RIS(1) condition.

By the definition of the norm, we then get that

$$\Big\| \sum_{i=1}^k U w_i \Big\| \geqslant f(k)^{-1/2} \sum_{i=1}^k w_i^*(U w_i) > \epsilon k f(k)^{-1/2} \ .$$

On the other hand, we can let g be the function given by Lemma 6 in the case $K_0 = K \setminus \{k\}$ and apply Lemma 3 to get an upper bound for $\|w_1 + \ldots + w_k\|$ of $24 k f(k)^{-1}$. This shows that $\|U\| > (\epsilon/24) f(k)^{1/2}$ for every $k \in K$, contradicting the continuity of T.

To do this, let $w = (1/8)(w_1 + \ldots + w_k)$. By Lemma 3, it is enough to show that, given any special functional $u^* = f(k)^{-1/2}(u_1^* + \ldots + u_k^*)$ of size k, and any interval E, we have $|u^*|(|Ew|) < 1$. Let t be maximal such that $u_t^* = w_t^*$. Then, for $i \leqslant t$ we have, by the conditions on the supports of the y_n and z_n, that $u_i^*(w_j) = 0$ for every j. If $i > t+1$, then $u_i^* \in A_N^*$ for some $N \in L$ distinct from all of $N_1, \ldots, N_k$. This gives a good upper bound for $|u_i^*|(|w_j|)$. If $N < N_j$ then $N < \sqrt{N_j}$ by choice of L. Since $\|\,|w_j|\,\|_{(\sqrt{N_j})} \leqslant 8$, we have

$|u_i^*|(|w_j|) \leqslant 8f(N)^{-1}$. If $N > N_j$ then $N >> N_j$ and, by Lemma 1, $|u_i^*|(|w_j|) \leqslant 3f(N_j)^{-1}$. These bounds are certainly good enough to give $|u^*|(|Ew|) \leqslant 1$ for every such u^* and finish the proof. $\square$

Corollary 28. *If $(x_n)_{n=1}^\infty \in \mathcal{L}$ then $Tx_n \to 0$.*

Proof. Let $A_n = \operatorname{supp}(x_n)$ as before. Suppose $|A_n|$ is even. Then because $\operatorname{diag}(T) = 0$, we have that Tx_n is four times the average of all vectors of the form $C_nTB_nx_n$, where $B_n \cup C_n = A_n$ and $|B_n| = |C_n| = |A_n|/2$. It follows that $\|Tx_n\|$ is at most four times the maximum of $\|C_nTB_nx_n\|$, which converges to zero by Lemma 27. If $|A_n|$ is odd, only a small modification is needed. One can average over $C_nTB_nx_n$ where $B_n \cup C_n$ is a partition of A_n and $|B_n|$ and $|C_n|$ differ by at most 1. Then four above must be replaced by $4n^2/(n^2-1)$. $\square$

Theorem 29. *Let T be any operator on the space X. Then $T - \operatorname{diag} T$ is strictly singular.*

Proof. The previous corollary shows that $(T - \operatorname{diag} T)x_n \to 0$ for every $(x_n)_{n=1}^\infty \in \mathcal{L}$. But, by Lemma 4, this implies that $T - \operatorname{diag} T$ is strictly singular. $\square$

Corollary 30. *The space X is not isomorphic to any proper subspace of itself.*

Proof. Let T be an isomorphism on to some subspace of X. If zero were infinitely singular for $\operatorname{diag} T$, then it would also be for T, by Theorem 29. Hence the diagonal entries of the matrix of T are eventually bounded below in modulus. It follows that $\operatorname{diag} T$ is Fredholm. Moreover, both the rank and corank of $\operatorname{diag} T$ are just the number of zeros on the diagonal, so it has index zero. Therefore, by Theorem 29, T is Fredholm with index zero, so the subspace must be the whole of X. $\square$

(5.2) In this subsection we suggest possibilities for further work along the lines of this paper and we make some remarks. Some people may object to our new prime Banach space on the grounds that it has no non-trivial complemented subspaces. Being prime under these circumstances is not such a great achievement. One possible answer to this objection is to use Theorem 5 for a slightly larger proper set than the one we used in (4.2). There are various possibilities. One promising one is to let $\mathcal{S}$ be the set of all spreads $S_{A,B}$, where A and B are infinite arithmetic progressions in $\mathbb{N}$. We do not have a proof that this space is prime, but it looks likely.

The result of (5.1) is sufficiently similar to the applications of Theorem 5 to suggest that Theorem 5 can be generalized. It would be nice, for example, to avoid the technical restriction on proper sets (to do with pairs of integers). It can certainly be relaxed somewhat, but this was not necessary for us. A motivation for carrying out such a generalization is that it ought then to be possible to give further interesting examples of Banach spaces with an unconditional basis. For example, suppose one takes $\mathcal{S}$ to be the semigroup generated by the proper set from (4.4) and all diagonal maps with ± 1 down the diagonal. It seems likely that the resulting space $X(\mathcal{S})$, a sort of combination of the spaces from (4.4) and (5.1), would be a space with an unconditional basis isomorphic to its cube but not its square.

Such a space together with its square would be the first example of a solution to the Schroeder-Bernstein problem using spaces with an unconditional basis. Recall that the Schroeder-Bernstein problem for Banach spaces asks whether, if X and Y are two Banach spaces isomorphic to complemented subspaces of each other, they must be isomorphic. A counterexample was given in [G2]. The spaces in (4.3) and (4.4) also produce counterexamples.

Another possibility would be to take $\mathcal{S}$ to be the semigroup generated by all spreads and all ± 1-diagonal matrices. This might well be a prime Banach space with an unconditional basis, which would surely be a genuine example by any standards.

Going back to proper sets of spreads, note that for any such set $\mathcal{S}$, the space $X(\mathcal{S})$ has no unconditional basic sequence. To see this, suppose that Y is a subspace generated by an unconditional block basis. Then $L(Y, X)$ is not separable, even in the $\|\!|.|\!\|$-norm. Since $\mathcal{S}$ is countable, this contradicts Theorem 5. It follows from [G3] that $X(\mathcal{S})$ has a hereditarily indecomposable subspace. In other words, the extra structure given to these spaces by $\mathcal{S}$ disappears when one passes to an appropriate subspace. It can in fact be shown that for a proper set $\mathcal{S}$ the space $X(\mathcal{S})$ has a hereditarily indecomposable subspace generated by a subsequence of the unit vector basis.

References.

[B] B. Blackadar, *K-theory for operator algebras*, MSRI Publications 5, Springer Verlag 1986.

[C] J. Cuntz, *K-theory for certain* C**-algebras* Ann. of Math. **113** (1981), 181-197.

[G1] W. T. Gowers, *A solution to Banach's hyperplane problem*, Bull. L.M.S., to appear.

[G2] W. T. Gowers, *A solution to the Schroeder-Bernstein problem for Banach spaces*, submitted.

[G3] W. T. Gowers, *A new dichotomy for Banach spaces*, preprint.

[GM] W. T. Gowers and B. Maurey, *The unconditional basic sequence problem*, Journal A.M.S. **6** (1993), 851-874.

[L] J. Lindenstrauss, *On complemented subspaces of m*, Israel J. Math. **9** (1971), 279-284.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I: sequence spaces*, Springer Verlag (1977).

[P] A. Pełczyński, *Projections in certain Banach spaces*, Studia Math. 19 (1960), 209-228.

[S] T. Schlumprecht, *An arbitrarily distortable Banach space*, Israel J. Math. **39** (1991), 81-95.