M 2003 **NUMBERS AND SETS – EXAMPLES SHEET 2** W. T. G.

1. Prove by induction that the following two statements are true for every positive integer n.

(i) The number $2^{n+2} + 3^{2n+1}$ is a multiple of 7.

(ii) $1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3 = n^2(2n^2 - 1)$.

2. Suppose that you have a $2^n \times 2^n$ grid of squares (if $n = 3$ then you have a chessboard) and you remove one square. Prove that, wherever the removed square is, the remaining squares can be tiled with L-shaped tiles - that is, tiles consisting of three squares that form a 2×2 grid with one square removed.

3. By considering the equation $(1-1)^n = 0$, give another proof that exactly half the subsets of $\{1, 2, \ldots, n\}$ have even size.

4. Prove that

$$\binom{k}{k} + \binom{k+1}{k} + \ldots + \binom{n+k-1}{k} = \binom{n+k}{k+1}$$

for any $n > k$. [Hint: for each set of size $k+1$ consider its largest element.]

5. Prove that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n} .$$

[Hint: $\binom{n}{k} = \binom{n}{n-k}$.]

6. There are four primes between 0 and 10 and four between 10 and 20. Does it ever happen again that there are four primes between two consecutive multiples of 10?

7. Is $n^2 + n + 41$ prime for all positive integers n?

8. Does there exist a block of 100 consecutive positive integers, none of which is prime?

9. Is there a power of 2 that begins with a 7?

10. Write down carefully (while not looking at your notes) a proof that there are infinitely many primes. By considering numbers of the form $4p_1p_2 \ldots p_k - 1$, prove that there are

infinitely many primes of the form $4n-1$. What would go wrong if we tried a similar proof to show that there are infinitely many primes of the form $4n+1$?

11. Translate the following sentence into a short English one, and write down its negation in symbolic form. (For this question the letters m, n, a, b should be understood as ranging over all positive integers - for instance, $\forall m$ really means $\forall m \in \mathbb{N}$.)

$$\forall m \; \exists n \; \forall a \; \forall b \; (n \geq m) \wedge \big[(a=1) \vee (b=1) \vee \big((ab \neq n) \wedge (ab+2 \neq n)\big)\big]$$

12. Find the highest common factor of 12345 and 54321.

13. Find integers x and y with $76x + 45y = 1$. Do there exist integers x and y with $1992x + 1752y = 12$?

14. Prove that if a is coprime to b and also to c then it is coprime to bc. Give two proofs: one based on Bezout's theorem and one based on prime factorisation.

15. Is it true that for all positive integers a, b, c, d we have $(a, b)(c, d) = (ac, bd)$?

16. Show that a positive integer n is a multiple of 9 if and only if the sum of its digits is a multiple of 9. Find a rule for when a number is a multiple of 11 and prove that your rule works.

17. The *Fibonacci numbers* $F_1, F_2, F_3, \ldots$ are defined by: $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n > 2$ (so eg. $F_3 = 2$, $F_4 = 3$, $F_5 = 5$). Is F_{2004} even or odd? Is it a multiple of 3?

18. Suppose that we have some positive integers (not necessarily distinct) whose sum is 100. How large can their product be?

19. We are given a binary operation $*$ on the positive integers, satisfying
(i) $1 * n = n + 1$ for all n
(ii) $m * 1 = (m-1) * 2$ for all $m > 1$
(iii) $m * n = (m-1) * (m * (n-1))$ for all $m, n > 1$.
Find the value of $5 * 5$.

20. The *repeat* of a positive integer is obtained by writing it twice in a row (so for example the repeat of 254 is 254254). Is there a positive integer whose repeat is a perfect square?