Numbers and Sets - exercises for enthusiasts 2. W. T. G.

1. Does there exist an uncountable family $\mathcal{B}$ of subsets of $\mathbb{N}$ such that for every $A, B \in \mathcal{B}$ (distinct) the intersection of A with B is finite?

2. Is it possible to write the closed interval $[0,1]$ as a countably infinite union of disjoint closed (non-empty) subintervals?

3. Let f be a function from $\mathbb{R}^2$ to $\mathbb{R}$ such that, for every x and y in $\mathbb{R}$, the functions $z \mapsto f(x,z)$ and $w \mapsto f(w,y)$ are polynomials. Prove that f is a polynomial in x and y. Does the result hold if $\mathbb{R}$ is replaced by $\mathbb{Q}$?

4. Let X be a non-empty set with an associative binary operation defined on it (which is written as multiplication in what follows). Suppose that for every $x \in X$ there is a unique x^* such that $xx^*x = x$. Prove that X is a group.

5. Is there a field that can be made into an ordered field in exactly three ways?

6. Given n points in the plane, not all collinear, show that it is possible to find a line containing exactly two of them.

7. Let G and H be groups such that $G \times \mathbb{Z} \equiv H \times \mathbb{Z}$. Show that G and H need not be isomorphic, but that they must if they are Abelian.

8. If X is a set, let $X^{(r)}$ denote the set of subsets of X of size r. Show that if $\mathbb{N}^{(r)}$ is coloured with finitely many colours, then there is an infinite subset $X \subset \mathbb{N}$ such that all the sets in $X^{(r)}$ have the same colour.

9. Call a set X of positive integers a *multiplicative basis of order* 2 if every positive integer n can be written as xy with x and y in X. Suppose that X is a multiplicative basis of order 2. Prove that there is a positive integer m that can be written as xy in at least 2003 different ways, all with x and y in X.

10. Let $..., a_{-1}, a_0, a_1, a_2, \ldots$ be a doubly infinite sequence of points in $[0,1]$ such that for every i, a_{i+1} is either $a_i + \sqrt{2}$ (mod 1) or $a_i + \sqrt{3}$ (mod 1). Can the set $A = \{a_i : i \in \mathbb{Z}\}$ be closed? (That means that any point not in A is contained in an open interval disjoint from A.)

11. Given a real number x, write $\log^2 x$ for $\log\log x$, $\log^3 x$ for $\log\log\log x$ and so on. Define a function f on real numbers greater than or equal to 1 by $f(x) = x(\log x)(\log^2 x)\ldots(\log^k x)$, where k is the largest integer for which $\log^k x$ is greater than or equal to 1. (If $k = 0$ then $f(x)$ is interpreted to be x.) Determine whether the sum $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converges or diverges.