MATHEMATICAL TRIPOS PART III (2025–26)

Local Fields - Example Sheet 3 of 3 **T.A. Fisher**

Except where stated otherwise: K is a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}_K$, normalised discrete valuation v_K, uniformiser π_K, and residue field k. We write μ_n for the group of nth roots of unity, and ζ_n for a primitive nth root of unity.

1. Compute the ramification groups of $\mathbb{Q}_3(\zeta_3, \sqrt[3]{2})/\mathbb{Q}_3$.

2. Prove that $\mathbb{Q}_2$ has a unique Galois extension with Galois group $(\mathbb{Z}/2\mathbb{Z})^3$. Compute its ramification groups.

3. Prove that there is at most one prime p for which $\mathbb{Q}_p$ has a Galois extension with Galois group S_4. If you like, you can try to construct such an extension.

4. Determine $\mathrm{Gal}(\mathbb{Q}_p(\zeta_8)/\mathbb{Q}_p)$ for every prime p.

5. Let $K = \mathbb{Q}_p(\zeta_p)$. Show that $(1-\zeta_p^i)/(1-\zeta_p) \equiv i \pmod{\pi_K}$ for all $1 \leqslant i \leqslant p-1$, and that $(1-\zeta_p)^{p-1} = -pu$ for some $u \in 1+\pi_K\mathcal{O}_K$. Deduce that $\mathbb{Q}_p(\zeta_p) = \mathbb{Q}_p(\sqrt[p-1]{-p})$.

6. Prove that $\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p$ is a totally ramified Galois extension, determine its degree, its Galois group and all the ramification groups G_i. (Hint: $1-\zeta_{p^n}$ is a uniformiser.)

7. Suppose L/K is a Galois, totally and tamely ramified extension of degree n. Prove that $\mu_n \subset K$ and $L = K(\sqrt[n]{\pi_K})$ for some choice of uniformiser π_K. How many totally and tamely ramified Galois extensions does $\mathbb{Q}_5$ have? (Hint: You may use Kummer's theorem: suppose k is any field of characteristic prime to n, containing μ_n. Then every cyclic Galois extension of degree n of k is of the form $k(\sqrt[n]{\alpha})$ for some $\alpha \in k$.)

8. Determine $\mathrm{Gal}(\overline{K}/K)$ for $K = \mathbb{C}((t))$.

9. Let $L = K(\zeta_m)$ where m is an integer coprime to p. Let $g(X)$ be the minimal polynomial of ζ_m over K. Use a version of Hensel's lemma (see Example Sheet 2, Question 5) to show that $\overline{g} \in k[X]$ is irreducible. Deduce that L/K is unramified.

10. (i) Let L_1/K and L_2/K be finite extensions of K, at least one of which is Galois, with ramification indices e_1 and e_2. Suppose that $(e_1, e_2) = 1$. Show that L_1L_2/K has ramification index e_1e_2.

    (ii) Compute the valuation rings of $\mathbb{Q}_p(\zeta_p, \sqrt[p]{p})$ and $\mathbb{Q}_p(\zeta_p, \sqrt[2p-1]{p})$.

11. Let L/K be an unramified extension of degree n, and let $S \subset \mathcal{O}_K$ be a set of coset representatives for k. Show that if $\alpha \in \mathcal{O}_L$ with $k_L = k(\overline{\alpha})$ then every $y \in \mathcal{O}_L$ can be written uniquely in the form

$$y = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{n-1} \lambda_{ij}\alpha^j \right) \pi_K^i$$

for some $\lambda_{ij} \in S$. Deduce that $\mathcal{O}_L = \mathcal{O}_K[\alpha]$.

12. Use the theory of the Herbrand quotient (for G a cyclic group of order n acting trivially on K^*) to show that

$$|K^*/(K^*)^n| = \frac{n|\mu_n(K)|}{|n|_K}$$

where $\mu_n(K)$ is the set of nth roots of unity in K.

13. (i) Show that if L/K is finite then $N_{L/K}(L^*) \subset K^*$ is an open subgroup.

 (ii) Show that if $K = \mathbb{Q}_p$ and $L = \mathbb{Q}_p(\zeta_m)$ then

$$N_{L/K}(L^*) = \begin{cases} \langle p, 1 + p^n\mathbb{Z}_p \rangle & \text{if } m = p^n, \\ \langle p^f, \mathbb{Z}_p^* \rangle & \text{if } m = p^f - 1. \end{cases}$$

 [Hint: For $p \neq 2$ we know that $\mathbb{Z}_p^* \cong (\mathbb{Z}/p\mathbb{Z})^* \times \mathbb{Z}_p$ and so $1 + p^n\mathbb{Z}_p$ is the only subgroup of $\mathbb{Z}_p^*$ of index $p^{n-1}(p-1)$.]

 (iii) (Local version of the Kronecker-Weber theorem.) Deduce by local class field theory that if $K/\mathbb{Q}_p$ is abelian then $K \subset \mathbb{Q}_p(\zeta_d)$ for some d.

14. Let $Q(x, y, z) = ax^2 + by^2 + cz^2$ where a, b, c are non-zero integers with abc odd and square-free. Show that Q is soluble over the rationals if and only if

 (i) a, b, c do not all have the same sign, and

 (ii) a, b, c are not all congruent mod 4, and

 (iii) $-bc$ is a square mod p for all primes p dividing a, and likewise under all permutations of a, b, c.

15. Consider the equation $3x^3 + 4y^3 + 5z^3 = 0$. Assume that there are non-trivial solutions over $\mathbb{F}_p$ for all primes $p \geqslant 7$. (This can be proved using the theory of elliptic curves.) Show that there are non-trivial solutions over $\mathbb{Q}_p$ for every prime p.