MATHEMATICAL TRIPOS PART III (2025–26)

Local Fields - Example Sheet 1 of 3 **T.A. Fisher**

Note: the p-adic absolute value on $\mathbb{Q}$ is normalised so that $|p|_p = 1/p$.

1. Prove the *product formula* for absolute values on $\mathbb{Q}$: If $x \in \mathbb{Q}^*$ then
$$\prod_{\alpha} |x|_\alpha = 1$$
where $\alpha \in \{\infty, 2, 3, 5, \ldots\}$ lists all the normalised absolute values on $\mathbb{Q}$. (The analogous formula for number fields will appear on the next example sheet.)

2. If $c \in \mathbb{Z}_p$ satisfies $|c|_p < 1$ show that $(1+c)^{-1} = 1 - c + c^2 - c^3 + \ldots$ Hence or otherwise find $a \in \mathbb{Z}$ such that $|4a-1|_5 \leqslant 5^{-10}$.

3. (i) Show that the inclusion $\mathbb{Z} \subset \mathbb{Z}_p$ induces an isomorphism $\mathbb{Z}/p^n\mathbb{Z} \cong \mathbb{Z}_p/p^n\mathbb{Z}_p$ for all $n \geqslant 1$.
   (ii) Show that $\#(\mathbb{Z}_p/m\mathbb{Z}_p) = |m|_p^{-1}$ for all $m \in \mathbb{Z}_p$.
   (iii) Show that a subgroup of $\mathbb{Z}_p$ is open if and only if it has finite index.

4. Suppose $a \in \mathbb{Z}$ with $(a,p) = 1$. Prove that the sequence $(a^{p^n})_{n \geqslant 0}$ converges in $\mathbb{Q}_p$ and its limit is a $(p-1)^{\text{th}}$ root of unity which is congruent to a mod p.

5. (i) Show that $\mathbb{Z}[\frac{1}{p}]$ is dense in $\mathbb{Q}_p$.
   (ii) Recall that $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the group of all roots of unity in $\mathbb{C}$. Show that $\mathbb{Q}_p/\mathbb{Z}_p$ is isomorphic to the group of all p-power roots of unity.

6. (i) Prove that any $x \in \mathbb{Q}_p$ can be written (uniquely) in the form $x = \sum_{n=n_0}^{\infty} a_n p^n$ where $n_0 \in \mathbb{Z}$ and each $a_n \in \{0, 1, \ldots, p-1\}$.
   (ii) Show that $x \in \mathbb{Q}$ if and only if the sequence $(a_n)_n$ is eventually periodic.
   (iii) Let $s_p(n)$ denote the sum of the p-adic digits of $n \in \mathbb{Z}$. Prove the formula $v_p(n!) = (n - s_p(n))/(p-1)$.

7. Show that the equation $x^3 - 3x + 4 = 0$ has a unique solution in $\mathbb{Z}_7$, but has no solutions in $\mathbb{Z}_5$ or in $\mathbb{Z}_3$. How many are there in $\mathbb{Z}_2$?

8. Consider the series
$$\text{“}\sqrt{1+15}\text{”} = 1 + \sum_{n=1}^{\infty} \binom{1/2}{n} 15^n$$
where $\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}$. Show that the series converges to 4 with respect to the 3-adic absolute value, to -4 with respect to the 5-adic absolute value, and diverges with respect to all other absolute values on $\mathbb{Q}$.

9. Let K be a field that is complete with respect to a non-trivial absolute value $|\cdot|$. Show that K is uncountable. (We know that if $|\cdot|$ is archimedean, then K contains the reals, so if you like you may assume $|\cdot|$ is non-archimedean.)

10. Let $\widehat{K}$ be the completion of a field K with respect to a valuation v. Show that v extends to a valuation $\widehat{v}$ on $\widehat{K}$. Prove that the value groups of v and $\widehat{v}$ are the same. In particular this shows that if v is discrete then $\widehat{v}$ is discrete.

11. Let k be an algebraically closed field and $K = k(t)$. Prove that every normalised discrete valuation on K which is trivial on k (i.e. $v(a) = 0$ for all $a \in k^*$) is either of the form v_a for some $a \in k$ ("order of vanishing at a") or is $v_\infty(p/q) = \deg q - \deg p$. What happens if k is not algebraically closed?

12. Let $\mathbb{Z}[[T]]$ be the ring of formal power series over $\mathbb{Z}$. Show that there is an isomorphism of rings $\mathbb{Z}[[T]]/(T-p) \cong \mathbb{Z}_p$ induced by the natural ring homomorphism
$$\mathbb{Z}[[T]] \to \mathbb{Z}_p\,; \quad \sum a_n T^n \mapsto \sum p^n a_n.$$

13. (Another form of Hensel's lemma) Let $f_1, \ldots, f_r \in \mathbb{Z}_p[X_1, \ldots, X_d]$ with $r \leqslant d$. Suppose that $a = (a_1, \ldots, a_d) \in \mathbb{Z}_p^d$ with $f_i(a) \equiv 0 \pmod{p}$ for all $1 \leqslant i \leqslant r$ and $\operatorname{rank}(\frac{\partial f_i}{\partial x_j}(a) \mod p) = r$. Show that there exists $x \in \mathbb{Z}_p^d$ with $x \equiv a \pmod{p}$ and $f_i(x) = 0$ for all $1 \leqslant i \leqslant r$. Is x unique?

14. (Strassman's theorem) Let $f(T) = a_0 + a_1 T + a_2 T^2 + \ldots \in \mathbb{Z}_p[[T]]$ be a formal power series with $a_n \to 0$ as $n \to \infty$. Suppose that for some $N \geqslant 0$ we have $v_p(a_N) = 0$ and $v_p(a_n) > 0$ for all $n > N$. Show that $\#\{x \in \mathbb{Z}_p : f(x) = 0\} \leqslant N$.