1. Let $\mathcal{M}$ be a smooth manifold. For $k \geq 1$, let us define the sub-bundle $\Lambda^k(\mathcal{M}) \subset \bigotimes_{i=1}^k T^*(\mathcal{M})$ consisting of $A \in \bigotimes_{i=1}^k T^*(\mathcal{M})$ such that $A_{i_1,\dots,i_n,\dots,i_m,\dots,i_k} = -A_{i_1,\dots,i_m,\dots,i_n,\dots,i_k}$ for all $1 \leq n \leq m \leq k$. For $k = 0$, let $\Lambda^0(\mathcal{M})$ denote the trivial $\mathbb{R}$-bundle over $\mathcal{M}$. Compute the dimension of Λ^k. We call smooth sections of Λ^k differential forms of degree k, or k-forms. We denote $\Omega^k(\mathcal{M}) = \Gamma(\Lambda^k(\mathcal{M}))$. Show that a smooth map of manifolds $f : \mathcal{M} \to \mathcal{N}$ defines a map $f^* : \Omega^k(\mathcal{N})) \to \Omega^k(\mathcal{M})$ called the pullback.

2. Define a natural map

$$d : \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})$$

by

$$(d\omega)_{i_1,\dots,i_{k+1}} = \frac{1}{k!}\sum_{\pi} \mathrm{sign}(\pi)\partial_{i_{\pi(1)}}\omega_{i_{\pi(2)},\dots,i_{\pi(k+1)}},$$

where π ranges over permutations of $1,\dots k+1$. Show that $d \circ d = 0$. Show that $d : \Omega^0 \to \Omega^1$ corresponds to the map already defined. Show that d commutes with the pullback operation, i.e. if $f : \mathcal{M} \to \mathcal{N}$ is a smooth map of manifolds, and ω is a form on $\mathcal{N}$ then $f^*(d\omega) = df^*(\omega)$.

3. We call a k-form ω closed if $d\omega = 0$. We call ω exact if there exists a $k-1$-form η such that $d\eta = \omega$. Show that an exact form is necessarily closed. For each $k \geq 0$ define an equivalence relation on the set of closed forms in Ω^k by $\omega_1 \sim \omega_2$ when there exists an η such that $d\eta = \omega_1 - \omega_2$. Show that this is indeed an equivalence relation. Denote the set of such equivalence classes by $H^k(\mathcal{M})$. Show that the $H^k(\mathcal{M})$ are in fact vector spaces. The H^k define the so-called de Rham cohomology of $\mathcal{M}$. Show that if f is a smooth map $f : \mathcal{M} \to \mathcal{N}$, then f induces a linear map $f^* : H^k(\mathcal{N}) \to H^k(\mathcal{M})$.

4. Given ω_1, ω_2 k_1 and k_2-forms, respectively, define the $k = k_1 + k_2$-form $\omega_1 \wedge \omega_2$ by

$$(\omega_1 \wedge \omega_2)_{i_1,\dots,i_k} = \frac{1}{k_1!k_2!}\sum_{\pi} \mathrm{sign}(\pi)\omega_{i_{\pi(1)},\dots i_{\pi(k_1)}}\omega_{i_{\pi(k+1)}\dots i_{\pi(k)}},$$

where π ranges over permutations of $1,\dots k$. Show that this indeed defines a k-form. Show moreover that $\wedge$ induces on $H^*(\mathcal{M}) = \bigoplus_{i=1}^{\infty} H^i(\mathcal{M})$ the structure of an associative algebra. Show that $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge d\omega_2$. Show that if $f : \mathcal{M} \to \mathcal{N}$ is a smooth map of manifolds, then $f^*(\omega_1 \wedge \omega_2) = f^*(\omega_1) \wedge f^*(\omega_2)$. Finally, show that $\wedge$ "descends" to cohomology, i.e. for $[\omega_1] \in H^{k_1}$, $[\omega_2] \in H^{k_2}$, represented by $\omega_1 \in \Omega^{k_1}$, $\omega_2 \in \Omega^{k_2}$, we have $[\omega_1 \wedge \omega_2] = [\omega_1] \wedge [\omega_2]$. Show that if $f : \mathcal{M} \to \mathcal{N}$ is smooth, then $f^* : H^*(\mathcal{N}) \to H^*(\mathcal{M})$ is an algebra homomorphism.

5. Let $\mathcal{M}$ be a compact n-dimensional manifold and let η be an n-dimensional form. We define $\int_{\mathcal{M}} \eta$ as follows: Let $\{\mathcal{U}_i, \phi_i\}$ define a finite atlas, and let χ_i define a subordinate partition of unity. Define $\int_{\mathcal{M}} \eta = \sum_i \int_{\mathcal{U}_1} (\phi_i^*\eta)_{12\cdots n} dx^1 \cdots dx^n$. Check that this definition does not depend on the choice of partition of unity. Finally, let ω be an $n-1$ form such that $d\omega = \eta$. Show that

$$\int_{\mathcal{M}} \eta = 0.$$

6. Let $(\mathcal{M}, g)$ be an n-dimensional Riemannian manifold. Let θ^i denote a collection of 1-forms such that $g^{kl}(\theta^i)_k(\theta^j)_l = 0$. We shall call such a collection an *orthonormal frame*. Show that one can always construct such a frame locally. Define now a collection of one forms ω^i_j, labelled by i and j ranging in $1 \dots n$, with $\omega^i_j = -\omega^j_i$, by

$$d\theta^i = -\omega^i_j \wedge \theta^j.$$

and define the collection of 2-forms Ω^i_j by

$$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j.$$

Show that Ω^i_j is related to the Riemann curvature tensor by the formula

$$\Omega^i_j(V,W)E_i = -R(V,W)E_i,$$

where V and W denote arbitrary vectors, and the collection of vector fields E_i is defined by $\theta^j(E_i) = \delta^j_i$. Use the above as a computational method to compute curvature for $\mathbb{S}^n$, $\mathbb{H}^n$, etc.

7. Let $\mathcal{M}$ be a smooth manifold of dimension n. An *orientation* on $\mathcal{M}$ is defined to be a global nowhere vanishing n-form, more precisely, an equivalence class of such n-forms, where $\omega \sim \tilde{\omega}$ if $\omega = f\tilde{\omega}$ for a smooth strictly positive function f. A manifold admitting an orientation is said to be *orientable*, and a manifold together with a particular orientation is said to be *oriented.* A smooth map $f : \mathcal{M} \to \mathcal{N}$ of oriented manifolds is said to be *orientation-preserving* if $f^*\omega$ is a representative for the orientation of $\mathcal{M}$ if ω is a representative for the orientation of $\mathcal{N}$. Show that the Möbius band defined on previous example sheets is not orientable.

8. Show that if $\mathcal{M}$ is a smooth n-dimensional compact orientable manifold, then $H^k(\mathcal{M})$ is finitely generated. Demonstrate a natural isomorphism $H^k(\mathcal{M}) \cong (H^{n-k}(\mathcal{M}))^*$. This is called *Poincaré duality.*

9. Let $\mathcal{M}$ be an orientable even-dimensional compact manifold with positive sectional curvature. Show that $\pi_1(\mathcal{M}) = \emptyset$. Hint: Argue by contradiction. If $\pi_1(\mathcal{M}) \neq \emptyset$, produce a closed geodesic minimizing arc length in its homotopy class. Now use a Jacobi field argument to produce a shorter homotopic curve.

10. Let $\mathcal{M}$ be an $n+1$-dimensional smooth manifold, for $n \geq 0$. A symmetric 2-tensor $g_{\alpha\beta}$ is said to define a *Lorentzian metric* if at each point p, there exists a basis $E_0, E_1, \ldots, E_n$ of $T_p\mathcal{M}$ such that $g(E_0,E_0) = -1$, $g(E_i,E_j) = \delta_{ij}$, $g(E_0,E_i) = 0$, where i, j range in $1 \ldots n$. We call $(\mathcal{M}, g)$ a *Lorentzian manifold.* Show that there exists a unique torsion-free symmetric connection compatible with g, and thus geodesics and the various curvature tensors can be defined exactly as in the Riemannian case.

11. Let $(\mathcal{M}, g)$ be a Lorentzian manifold, and let $v \in T_p\mathcal{M}$. We call v *timelike*, *null*, *spacelike*, according to whether $g(v,v) < 0$, $g(v,v) = 0$, $g(v,v) > 0$. We call a non-zero vector v *causal* if it is either timelike or null. A vector field V is called *timelike*, etc., if V_p is timelike for all p. A globally defined timelike vector field T on $\mathcal{M}$ defines a *time-orientation.* A Lorentzian manifold that admits a time-orientation is said to be *time-orientable.* Given a time-orientation T, we say that a causal vector v is future-directed if $g(v,T) > 0$ and past directed if $g(v,T) < 0$. We say that two globally defined timelike vector fields T and $\tilde{T}$ define the same time-orientation if $g(T,\tilde{T}) > 0$, i.e. a time orientation is actually an equivalence class $[T]$. A Lorentzian manifold with a given time-orientation is said to be *time-oriented.* Show that if $\mathcal{M}$ is time-orientable, then it is also orientable. Is the converse true? Show that timelike geodesics locally maximize the functional $L(\gamma) = \int \sqrt{-g(\gamma',\gamma')}dt$.

12. Let $(\mathcal{M}, g)$ be a Lorentzian manifold. A smooth submanifold $\mathcal{N} \subset \mathcal{M}$ is said to be timelike if the induced metric is Lorentzian, spacelike if the induced metric is Riemannian, and null if the induced metric is degenerate. If $\mathcal{N}$ has codimension 1, show that $\mathcal{N}$ is spacelike iff $\mathcal{N}$ can be locally expressed as $f = 0$, for a function whose gradient ∇f is timelike. On the other hand,

if $\mathcal{N}$ has dimension 1, show that it is timelike, etc. iff it can be parametrized by $\gamma : (0,1) \to \mathcal{M}$ such that γ' is timelike, etc.

13. Given $S \subset \mathcal{M}$, we define the *causal future* of S, denoted $J^+(S)$, by

$$J^+(S) = \{p \in \mathcal{M} : \exists \gamma : [0,1] \to \mathcal{M}, \gamma' \text{ future-directed, causal}, \gamma(0) \in S, \gamma(1) = p\},$$

where γ here always denotes a differentiable curve. We similarly define the *chronological future* of S, denoted $I^+(S)$, by replacing causal above by timelike. Finally, we define the causal and chronological pasts J^- and I^- by replacing future with past. Show that $I^+(S)$, $I^-(S)$ are open subsets of $\mathcal{M}$, and $I^+(I^+(S)) = I^+(\bar{S}) = I^+(S)$. Show that $J^+(S) \subset \overline{I^+(S)}$. Note: In general relativity, curves such that γ' is timelike correspond to physical observers. Timelike geodesics correspond to freely-falling observers. Null geodesics correspond to light rays in the geometric optics approximation.

14. Let $(\mathcal{M}, g)$ be a smooth time-oriented Lorentzian manifold. We often call such $\mathcal{M}$ *space-times*. We say that $\mathcal{M}$ is *globally hyperbolic* if there exists a smooth spacelike hypesurface Σ such that all inextendible causal curves intersect Σ once and only once. (An inextendible causal curve is a causal curve whose image is not a proper subset of another causal curve.) Show that a globally hyperbolic spacetime is necessarily non-compact. Show that if $(\mathcal{M}, g)$ is globally hyperbolic, then $J^+(K)$ is closed for all compact $K \in \mathcal{M}$. Show that this is not necessarily true in a non-globally hyperbolic spacetime.

15. Let Σ be a codimension-2 spacelike submanifold of a time-oriented Lorentzian manifold $(\mathcal{M}, g)$. Show that in a neighborhood of each point $p \in \Sigma$, we may define two future-directed null vector fields L, $\bar{L}$, such that $\mathrm{Span}(L, \bar{L}, T_p\Sigma) = T_p\mathcal{M}$, and such that L and $\bar{L}$ are both orthogonal to $T_p\Sigma$. Now for each X, Y in $\Gamma(T\Sigma)$, we may define $K(X,Y) = -g(\nabla_X L, Y)$, $\bar{K}(X,Y) = -g(\nabla_X \bar{L}, Y)$. Show that K and $\bar{K}$ are symmetric tensors on Σ in a neighborhood of p, in particular, they depend only on $X(p)$, $Y(p)$. Denoting local coordinates on Σ by x^i, we call Σ *trapped* if $g^{ij}K_{ij} < 0$, and $g^{ij}\bar{K}_{ij} < 0$. Show that the definition of trapped does not depend on the choice of L, $\bar{L}$.

16. Let Σ be a compact codimension-2 spacelike submanifold of a time-oriented globally hyperbolic Lorentzian manifold $(\mathcal{M}, g)$. Let γ be a future-directed null geodesic $\gamma : [0,T) \to \mathcal{M}$ such that $p = \gamma(0) \in \Sigma$. We say that a point $q = \gamma(t)$ is conjugate to Σ if there exists a Jacobi field J along γ such that $J(0)$ is orthogonal to $T_p\Sigma$ and $J(t) = 0$. Show that $J^+(\Sigma) \setminus I^+(\Sigma)$ is covered by null geodesics emanating from Σ, without conjugate points to Σ.

17. Let $(\mathcal{M}, g)$ be a time-oriented, globally hyperbolic Lorentzian manifold with Cauchy surface $\mathcal{S}$. Let Σ be a codimension-2 compact spacelike trapped submanifold of $\mathcal{M}$. Suppose that $\mathrm{Ric}(V,V) \geq 0$ for all null vectors V in $T\mathcal{M}$. Show that $\mathcal{M}$ is future-causally geodesically incomplete, i.e. that there exists an inextendible future-directed causal geodesic γ such that defining a parameter t such that $\nabla_{\gamma'}\gamma' = 0$, then γ is covered by the coordinate range $S < t < T$, for some $-\infty \leq S < T < \infty$. (This result is known as *Penrose's incompleteness theorem.*)

In general relativity, the assumption $\mathrm{Ric}(V,V) \geq 0$ holds for spacetimes solving the Einstein equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 2T_{\mu\nu}$ whenever $T_{\mu\nu}V^\mu V^\nu \geq 0$ for all null V. In particular, the condition clearly holds for the so-called vacuum equations where $T_{\mu\nu} = 0$!