1. Give an example of a Riemannian manifold not geodesically complete, but so that every p and q can be joined by a length-minimizing geodesic. Give an example of a Riemannian manifold for which any two points p and q can be joined by a geodesic, but for which there exist two points $\tilde{p}$ and $\tilde{q}$ which cannot be joined by a length-minimizing geodesic.

2. Recall the definition of the Riemann curvature tensor

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[Y,X]} Z. \tag{1}$$

Show the identities

$$\begin{aligned} R(X,Y)Z &= -R(Y,X)Z, \\ R(X,Y)Z + R(Y,Z)X + R(Z,X)Y &= 0, \\ g(R(X,Y)Z,W) &= g(R(X,W)Z,Y), \\ g(R(X,Y)Z,W) &= -g(R(X,Y)W,Z). \end{aligned}$$

Show that given now a vector bundle $E \to \mathcal{M}$, and a connection ∇, the definition (1) still makes sense, and $R \in \Gamma(T^*\mathcal{M} \otimes T^*\mathcal{M} \otimes E^* \otimes E)$. This is called the curvature tensor of the connection. Are the above identities (those that make sense, that is) still true?

3. Let $\mathbb{S}^n$ denote the standard n-sphere, and let $\mathbb{H}^n$ denote hyperbolic n-space. The latter is defined by the manifold $\{(x^1,\ldots,x^n) : (x^1)^2 + \cdots (x^n)^2 < 1\}$, with metric $g = -4((dx^1)^2 + \cdots (dx^n)^2)(1 - ((x^1)^2 + \cdots (x^n)^2))^{-1}$. Compute the Riemann curvature, sectional curvatures, Ricci curvature, and scalar curvature. Show that $\mathbb{H}^n$ is geodesically complete.

4. Let $\mathcal{N}$ denote a submanifold of codimension 1 of an n-dimensional Riemannian manifold $(\mathcal{M}, g)$. We define the second fundamental form, on a subset $\mathcal{U} \subset \mathcal{N}$, as follows: Let N denote a unit normal field on $\mathcal{U}$, i.e. a vector field defined along $\mathcal{U}$ such that $g(N,N) = 1$ and $g(N,T) = 0$ for all $T \in T_p\mathcal{N}$ for $p \in \mathcal{U}$. (There are two choices for N. Note which of the definitions that follow depend on the choice, and which do not.) For X, Y vector fields along $\mathcal{U}$, let $\tilde{X}$, $\tilde{Y}$, $\tilde{N}$, denote arbitrary extensions to a neighborhood $\tilde{\mathcal{V}}$ of $\mathcal{N}$ in $\mathcal{M}$, and define $B(X,Y) = -g(\nabla_{\tilde{X}} N, \tilde{Y})$. Show that this definition does not depend on the extensions. B is thus a covariant 2-tensor, i.e. an element of $\Gamma(T^*\mathcal{N} \otimes T^*\mathcal{N})$. Show moreover that B is symmetric.

5. Let $\mathcal{N}$, $(\mathcal{M}, g)$ be as above. Let B_{ij} denote (as usual) the components of the tensor B with respect to local coordinates on $\mathcal{N}$, and let $\bar{g}_{ij}$ denote the induced Riemannian metric on $\mathcal{N}$. Let $k_1 \ldots k_{n-1}$ denote the eigenvalues of B_{ij} with respect to g_{ij}. These are known as the *principal curvatures*. We call $\frac{1}{n-1}(k_1 + \cdots + k_{n-1}) = H$ the *mean curvature*. Show that $(n-1)H = g^{ij}B_{ij}$. We call $k_1 \cdot k_2 \cdots k_{n-1} = K$ the *Gauss curvature*. Show that if $n = 3$, $K = 2R$ where R is the scalar curvature of $(\mathcal{N}, \bar{g})$. This is the Theorema Egregium of Gauss. Derive a general relation relating the second fundamental form B_{ij} and the Riemann curvature tensor R^l_{kij}, valid in all dimensions $n \geq 2$.

6. For a connected Riemannian manifold, prove that metric completeness implies that every p and q can be joined by a length-minimizing geodesic by filling in the details to the following sketch: Let $\delta = d(p,q)$. Clearly, there exists a sequence of curves γ_i joining p and q such that $\lim L(\gamma_i) = \delta$. Extract a convergent subsequence of the γ_i. Argue that the limit of this sequence is the desired geodesic.

7. Now adapt the above argument to prove the following: Suppose $\mathcal{M}$ is compact. If $\pi_1(\mathcal{M})$ is nontrivial, then for each $[\gamma] \in \pi_1(\mathcal{M})$, there exists a $\tilde{\gamma} \in [\gamma]$ such that $\tilde{\gamma}$ is a closed geodesic, and $\tilde{\gamma}$ is length minimizing in its homotopy class, i.e. $L(\tilde{\gamma}) \leq L(\tilde{\gamma}')$ for all $\tilde{\gamma}' \in [\gamma]$. Does the existence of a closed geodesic in $(\mathcal{M}, g)$ imply that π_1 is non-trivial?

8. Now let $\mathcal{S}_1$ and $\mathcal{S}_2$ be smooth closed hypersurfaces in a connected Riemannian manifold $(\mathcal{M}, g)$. Suppose $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$, and suppose there exists an $\epsilon > 0$ and a compact set K, such that for all $p, q \in \mathcal{M} \setminus K$, $p \in \mathcal{S}_1$, $q \in \mathcal{S}_2$, we have $d(p,q) \geq \epsilon + d(\mathcal{S}_1, \mathcal{S}_2)$. Show that there exists a curve γ joining $\mathcal{S}_1$ and $\mathcal{S}_2$ which minimizes the distance between these two hypersurfaces. Show that γ is a geodesic orthogonal to both hypersurfaces. Now suppose conversely that γ is some geodesic connecting $\mathcal{S}_1$ and $\mathcal{S}_2$, which is orthogonal to both hypersurfaces. Show that γ locally extremizes the length functional for curves joining the hypersurfaces. Show by explicit example that γ is not necessarily a length minimizing curve. Investigate examples in Euclidean space.

9. Let $(\mathcal{M}, g)$ denote a Riemannian manifold, and let X, and Y be vector fields defined in a neighborhood of some point $p \in \mathcal{M}$, such that $[X, Y] = 0$ identically. For $t_0 \geq 0$, let A_{t_0} denote the parallel transport operator correponding to the curve $\gamma : [0, 4t_0] \to \mathcal{M}$ defined by

$$t \mapsto (\phi_t^X)(p), 0 \leq t \leq t_0$$

$$t \mapsto \phi_{t-t_0}^Y(\phi_{t_0}^X(p)), t_0 \leq t \leq 2t_0$$
$$t \mapsto \phi_{t-2t_0}^{-X}(\phi_{t_0}^Y(\phi_{t_0}^X(p))), 2t_0 \leq t \leq 3t_0$$
$$t \mapsto \phi_{t-3t_0}^{-Y}(\phi_{t_0}^{-X}(\phi_{t_0}^Y(\phi_{t_0}^X(p)))), 3t_0 \leq t \leq 4t_0,$$

where ϕ_t^X denotes the 1-parameter local group of transformations defined by X. We assume that t_0 is sufficiently small so the maps referred to above are defined. Note that $A_{t_0} : T_p\mathcal{M} \to T_p\mathcal{M}$. Show that

$$A_{t_0}(Z) = Id - t_0 R(X,Y)Z + o(t_0).$$

10. Recall the holonomy groups $\mathcal{G}_p$ of the previous example sheet. Use the above formula to show that for "generic" Riemannian metrics, $\mathcal{G}_p = SO(n)$. Justify your definition of genericity.