1. Let $\mathcal{M}$ be a smooth manifold, and ∇ a connection in $T\mathcal{M}$. Let ϕ_t define the 1-parameter group of local transformations of $T\mathcal{M}$ determining geodesic flow. Show that for any point $p \in \mathcal{M}$, ϕ_1 can be defined on an *open* set $\mathcal{U} \subset T\mathcal{M}$ containing 0_p. (It suffices to show that for the domain $\mathcal{U}$ of ϕ_1, there exists an open $\mathcal{W} \subset \mathcal{M}$ containing p, and $\epsilon > 0$, such that $\{V_q : |V_q| < \epsilon\}_{q\in\mathcal{W}} \subset \mathcal{U}$, where $|\cdot|$ denotes coordinate length.) We define the exponential map $\exp : \mathcal{U} \to \mathcal{M}$ by $\pi \circ \phi_1$, where $\pi : T\mathcal{M} \to \mathcal{M}$ denotes the natural projection. We call a connection ∇ geodesically complete if all solutions to $\nabla_T T = 0$ can be defined for all time, i.e. if the domain of exp can be taken as $T\mathcal{M}$. Show that compact manifolds are geodesically complete with respect to any connection ∇ in $T\mathcal{M}$.

2. Let $\pi : E \to \mathcal{M}$ be a vector bundle. Show that there exists a connection in E. Let ∇, $\tilde{\nabla}$ denote two connections in E. Show that $\nabla - \tilde{\nabla}$ defines a section of $T^*\mathcal{M} \otimes E^* \otimes E$.

3. Let $\mathcal{A} \subset \Gamma(T\mathcal{M})$ denote a Lie subalgebra, i.e. a subspace of $\Gamma(T\mathcal{M})$ closed under the Lie bracket. Let $p \in \mathcal{M}$. Show that there is a submanifold $\mathcal{N}$ of $\mathcal{M}$, such that $T_q\mathcal{N}$ is spanned by $\mathcal{A}$ for all q in a neighborhood of p in $\mathcal{N}$.

4. Let $\mathcal{M}$ be a smooth manifold, let $\pi : E \to \mathcal{M}$ be a vector bundle and let ∇, $\tilde{\nabla}$ denote connections on E. For any curve γ from p_1 to p_2, let $T^{\nabla}_{p_1,p_2,\gamma}$ denote the parallel transport map with respect to ∇. Suppose $T^{\nabla}_{p_1,p_2,\gamma} = T^{\tilde{\nabla}}_{p_1,p_2,\tilde{\gamma}}$ for all p_1, p_2, γ. Show that $\nabla = \tilde{\nabla}$, i.e., "parallel transport determines the connection". Give sufficient conditions on a collection of maps $\{T_{p_1,p_2,\gamma}\}$ such that these arise as parallel transport maps from a connection ∇.

5. Consider $\mathbb{E}^n$ with its standard Euclidean connection ∇. Let $\mathcal{M}$ denote a submanifold. Define a connection $\tilde{\nabla}$ on $\mathcal{M}$ by $\tilde{\nabla}_\xi Y = \pi_{T_p\mathcal{M}} \nabla_\xi Y$, where $Y \in \Gamma(T\mathcal{M})$, $\xi \in T_p\mathcal{M}$, and $\pi_{T_p\mathcal{M}} : T_p\mathbb{E}^n \to T_p\mathcal{M}$ denotes the projection. Show in particular that this definition can be made sense of even though $Y \notin \Gamma(T\mathbb{E}^n)$. Show that $\tilde{\nabla}$ coincides with the Levi-Civita connection of the induced metric g on $\mathcal{M}$.

6. Now let $\mathbb{S}^n$ denote the n-sphere in $\mathbb{E}^{n+1}$. Compute the Christoffel symbols of its induced Riemannian connection in your favourite system of local coordinates. Describe geometrically parallel transport. Are there vector fields defined in open subsets $\mathcal{U}$ of $\mathbb{S}^n$, $V : \mathcal{U} \to T\mathcal{M}$ such that $\nabla_\xi V = 0$ for all ξ? Prove that the sphere is not locally isometric to $\mathbb{R}^n$ using this information.

7. Let $\pi : E \to \mathcal{M}$ be a vector bundle, and let ∇ be a connection in E. Clearly, we can define the parellel transport map $T_{p_1,p_2,\gamma}$ not just for smooth curves γ, but for piecewise smooth curves. For $p \in \mathcal{M}$ consider the set $\{T_{p,p,\gamma}\}$, where γ ranges over all piecewise smooth closed curves from p to p. Show that this defines a subgroup $\mathcal{G}_p$ of the group of linear isomorphisms of V_p, and explain the composition law. This is known as the *holonomy group* at p. Show that if $\mathcal{M}$ is connected then $\mathcal{G}_p \cong \mathcal{G}_q$. Let ∇ now be the Levi-Civita connection of a Riemannian metric g. Show that $\mathcal{G}_p$ is a subgroup of the group of isometries of the tangent space.

8. Compute $\mathcal{G}_p$ for $\mathbb{R}^n$ and for $\mathbb{S}^n$ with their Levi-Civita connections. Produce a connected n-dimensional Riemannian manifold whose Levi-Civita connection gives a holonomy group bigger than that of $\mathbb{R}^n$ and smaller than that of $\mathbb{S}^n$. Produce a simply-connected such manifold.

9. Recall the vector bundle E defining the Möbius strip and let ∇ be a connection on E. What are the possibilities for $\mathcal{G}_p$?

10. Fill in the gaps in the proof of the first variation formula, i.e. in the proof that a curve $\gamma_0(t) = \gamma(t, 0)$ extremizes length over variations $\gamma(t, s)$ with the same fixed points iff $\nabla_{\gamma_* \frac{\partial}{\partial t}} \gamma_* \frac{\partial}{\partial t} = 0$. In particular, justify why from

$$\int_a^b (\nabla_{\gamma_* \frac{\partial}{\partial t}} \gamma_* \frac{\partial}{\partial t}\big|_{s=0}, \gamma_* \frac{\partial}{\partial s}\big|_{s=0}) dt = 0$$

for all variations $\gamma(t, s)$, we can infer that $\nabla_{\gamma_* \frac{\partial}{\partial t}} \gamma_* \frac{\partial}{\partial t}\big|_{s=0} = 0$ for all t.

11. Let $(\mathcal{M}, g)$ be connected, and locally isometric to $\mathbb{R}^n$. Show that if γ is a topologically trivial closed curve, then $T_{p,p}\gamma = id$. More generally, show that there exists a well-defined group homomorphism

$$\phi : \pi_1(\mathcal{M}) \to \mathcal{G}_p$$

where π_1 denotes the fundamental group of $\mathcal{M}$.

Show that a Riemannian manifold $(\mathcal{M}, g)$ is locally isometric to $\mathbb{R}^n$ iff for all p, there exits a $\mathcal{U}$ containing p such that $T_{p,p,\gamma} = id$ for all $\gamma \subset \mathcal{U}$. Show that the condition $\gamma \subset \mathcal{U}$ can be weakened to the condition that γ is topologically trivial.

12. Let $(\mathcal{M}^n, g)$ be Riemannian. If $n = 2$, show that the condition $\nabla_\xi g(X, Y) = g(\nabla_\xi X, Y) + g(X, \nabla_\xi Y)$, together with the requirement that length extremizing curves γ, parametrized by arc length, should satisfy $\nabla_{\gamma'}\gamma' = 0$, determines ∇ to be Levi-Civita. Show that this is no longer the case for $n > 2$.