

1. Let  $\mathcal{M}$  be a smooth manifold, and  $\nabla$  a connection in  $T\mathcal{M}$ . Let  $\phi_t$  define the 1-parameter group of local transformations of  $T\mathcal{M}$  determining geodesic flow. Show that for any point  $p \in \mathcal{M}$ ,  $\phi_1$  can be defined on an *open* set  $\mathcal{U} \subset T\mathcal{M}$  containing  $0_p$ . (It suffices to show that for the domain  $\mathcal{U}$  of  $\phi_1$ , there exists an open  $\mathcal{W} \subset \mathcal{M}$  containing  $p$ , and  $\epsilon > 0$ , such that  $\{V_q : |V_q| < \epsilon\}_{q \in \mathcal{W}} \subset \mathcal{U}$ , where  $|\cdot|$  denotes coordinate length.) We define the exponential map  $\exp : \mathcal{U} \rightarrow \mathcal{M}$  by  $\pi \circ \phi_1$ , where  $\pi : T\mathcal{M} \rightarrow \mathcal{M}$  denotes the natural projection. We call a connection  $\nabla$  geodesically complete if all solutions to  $\nabla_T T = 0$  can be defined for all time, i.e. if the domain of  $\exp$  can be taken as  $T\mathcal{M}$ . Show that compact manifolds are geodesically complete with respect to any connection  $\nabla$  in  $T\mathcal{M}$ .
2. Let  $\pi : E \rightarrow \mathcal{M}$  be a vector bundle. Show that there exists a connection in  $E$ . Let  $\nabla, \tilde{\nabla}$  denote two connections in  $E$ . Show that  $\nabla - \tilde{\nabla}$  defines a section of  $T^*\mathcal{M} \otimes E^* \otimes E$ .
3. Let  $\mathcal{A} \subset \Gamma(T\mathcal{M})$  denote a Lie subalgebra, i.e. a subspace of  $\Gamma(T\mathcal{M})$  closed under the Lie bracket. Let  $p \in \mathcal{M}$ . Show that there is a submanifold  $\mathcal{N}$  of  $\mathcal{M}$ , such that  $T_q\mathcal{N}$  is spanned by  $\mathcal{A}$  for all  $q$  in a neighborhood of  $p$  in  $\mathcal{N}$ .
4. Let  $\mathcal{M}$  be a smooth manifold, let  $\pi : E \rightarrow \mathcal{M}$  be a vector bundle and let  $\nabla, \tilde{\nabla}$  denote connections on  $E$ . For any curve  $\gamma$  from  $p_1$  to  $p_2$ , let  $T_{p_1, p_2, \gamma}^\nabla$  denote the parallel transport map with respect to  $\nabla$ . Suppose  $T_{p_1, p_2, \gamma}^\nabla = T_{p_1, p_2, \tilde{\gamma}}^{\tilde{\nabla}}$  for all  $p_1, p_2, \gamma$ . Show that  $\nabla = \tilde{\nabla}$ , i.e., “parallel transport determines the connection”. Give sufficient conditions on a collection of maps  $\{T_{p_1, p_2, \gamma}\}$  such that these arise as parallel transport maps from a connection  $\nabla$ .
5. Consider  $\mathbb{E}^n$  with its standard Euclidean connection  $\nabla$ . Let  $\mathcal{M}$  denote a submanifold. Define a connection  $\tilde{\nabla}$  on  $\mathcal{M}$  by  $\tilde{\nabla}_\xi Y = \pi_{T_p\mathcal{M}} \nabla_\xi Y$ , where  $Y \in \Gamma(T\mathcal{M})$ ,  $\xi \in T_p\mathcal{M}$ , and  $\pi_{T_p\mathcal{M}} : T_p\mathbb{E}^n \rightarrow T_p\mathcal{M}$  denotes the projection. Show in particular that this definition can be made sense of even though  $Y \notin \Gamma(T\mathbb{E}^n)$ . Show that  $\tilde{\nabla}$  coincides with the Levi-Civita connection of the induced metric  $g$  on  $\mathcal{M}$ .
6. Now let  $\mathbb{S}^n$  denote the  $n$ -sphere in  $\mathbb{E}^{n+1}$ . Compute the Christoffel symbols of its induced Riemannian connection in your favourite system of local coordinates. Describe geometrically parallel transport. Are there vector fields defined in open subsets  $\mathcal{U}$  of  $\mathbb{S}^n$ ,  $V : \mathcal{U} \rightarrow T\mathcal{M}$  such that  $\nabla_\xi V = 0$  for all  $\xi$ ? Prove that the sphere is not locally isometric to  $\mathbb{R}^n$  using this information.
7. Let  $\pi : E \rightarrow \mathcal{M}$  be a vector bundle, and let  $\nabla$  be a connection in  $E$ . Clearly, we can define the parallel transport map  $T_{p_1, p_2, \gamma}$  not just for smooth curves  $\gamma$ , but for piecewise smooth curves. For  $p \in \mathcal{M}$  consider the set  $\{T_{p, p, \gamma}\}$ , where  $\gamma$  ranges over all piecewise smooth closed curves from  $p$  to  $p$ . Show that this defines a subgroup  $\mathcal{G}_p$  of the group of linear isomorphisms of  $V_p$ , and explain the composition law. This is known as the *holonomy group* at  $p$ . Show that if  $\mathcal{M}$  is connected then  $\mathcal{G}_p \cong \mathcal{G}_q$ . Let  $\nabla$  now be the Levi-Civita connection of a Riemannian metric  $g$ . Show that  $\mathcal{G}_p$  is a subgroup of the group of isometries of the tangent space.
8. Compute  $\mathcal{G}_p$  for  $\mathbb{R}^n$  and for  $\mathbb{S}^n$  with their Levi-Civita connections. Produce a connected  $n$ -dimensional Riemannian manifold whose Levi-Civita connection gives a holonomy group bigger than that of  $\mathbb{R}^n$  and smaller than that of  $\mathbb{S}^n$ . Produce a simply-connected such manifold.
9. Recall the vector bundle  $E$  defining the Möbius strip and let  $\nabla$  be a connection on  $E$ . What are the possibilities for  $\mathcal{G}_p$ ?

10. Fill in the gaps in the proof of the first variation formula, i.e. in the proof that a curve  $\gamma_0(t) = \gamma(t, 0)$  extremizes length over variations  $\gamma(t, s)$  with the same fixed points iff  $\nabla_{\gamma_* \frac{\partial}{\partial t}} \gamma_* \frac{\partial}{\partial t} = 0$ . In particular, justify why from

$$\int_a^b (\nabla_{\gamma_* \frac{\partial}{\partial t}} \gamma_* \frac{\partial}{\partial t} \Big|_{s=0}, \gamma_* \frac{\partial}{\partial s} \Big|_{s=0}) dt = 0$$

for all variations  $\gamma(t, s)$ , we can infer that  $\nabla_{\gamma_* \frac{\partial}{\partial t}} \gamma_* \frac{\partial}{\partial t} \Big|_{s=0} = 0$  for all  $t$ .

11. Let  $(\mathcal{M}, g)$  be connected, and locally isometric to  $\mathbb{R}^n$ . Show that if  $\gamma$  is a topologically trivial closed curve, then  $T_{p,p}\gamma = id$ . More generally, show that there exists a well-defined group homomorphism

$$\phi : \pi_1(\mathcal{M}) \rightarrow \mathcal{G}_p$$

where  $\pi_1$  denotes the fundamental group of  $\mathcal{M}$ .

Show that a Riemannian manifold  $(\mathcal{M}, g)$  is locally isometric to  $\mathbb{R}^n$  iff for all  $p$ , there exists a  $\mathcal{U}$  containing  $p$  such that  $T_{p,p}\gamma = id$  for all  $\gamma \subset \mathcal{U}$ . Show that the condition  $\gamma \subset \mathcal{U}$  can be weakened to the condition that  $\gamma$  is topologically trivial.

12. Let  $(\mathcal{M}^n, g)$  be Riemannian. If  $n = 2$ , show that the condition  $\nabla_\xi g(X, Y) = g(\nabla_\xi X, Y) + g(X, \nabla_\xi Y)$ , together with the requirement that length extremizing curves  $\gamma$ , parametrized by arc length, should satisfy  $\nabla_{\gamma'} \gamma' = 0$ , determines  $\nabla$  to be Levi-Civita. Show that this is no longer the case for  $n > 2$ .