Lent Term 2009 J. M. E. Hyland

Set Theory and Logic: Example Sheet 1

1. In lectures I usually give subsets of the reals of order types $\omega + \omega$ and ω^2. Write down subsets of the reals with those order types and also one of order type ω^3.

2. (i) Show that a partial order $(A, <)$ is total if and only if any two initial segments are comparable under $\subseteq$.

   (ii) Suppose that A is a well-ordered set. Show that any subset $X \subseteq A$ is order isomorphic to an initial segment of A. Show conversely that if A be a totally ordered set such that every subset of A is order isomorphic to some initial segment of A, then A is a well-ordering.

3. (i) Let A be a well-ordering of order type α. Show that the set of initial segments of A is well-ordered by inclusion and determine its order type.

   (ii) Suppose that X is a set of ordinals with no maximal element. Show that $\sup X$ cannot be a successor ordinal.

4. For a well-ordered set A, write $\mu(A)$ for the corresponding (von Neumann if you wish) ordinal.

   (i) Suppose that A is a well-ordered set. Consider $A \supseteq X_0 \supseteq X_1 \supseteq X_2 \cdots$ a decreasing subsequence of subsets of A. Why is $\mu(X_n)$ eventually constant? Give an example to show that we do not necessarily have $\mu(\bigcap_n X_n) = \bigcap_n \mu(X_n)$?

   (ii) Suppose that A is a well-ordered set. Consider $X_0 \subseteq X_1 \subseteq X_2 \cdots \subseteq A$ an increasing subsequence of subsets of A. Give an example to show that we do not necessarily have $\mu(\bigcup_n X_n) = \bigcup_n \mu(X_n)$? What simple condition can you impose on the X_n to make this equation true?

5. What is the least ordinal α such that $1 + \alpha = \alpha$? The next least? The one after that?
   What is the least ordinal α such that $\omega + \alpha = \alpha$? The next least? The one after that?
   What is the least ordinal α such that $\omega.\alpha = \alpha$? The next least? The one after that?
   What about $\alpha.\omega = \alpha$?

6. (i) Prove that $\alpha.(\beta + \gamma) = \alpha.\beta + \alpha.\gamma$ and that $\alpha.(\beta.\gamma) = (\alpha.\beta).\gamma$.
   What about $(\alpha + \beta).\gamma = \alpha.\gamma + \beta.\gamma$?

   (ii) Establish the following properties of ordinal subtraction

   $$(\alpha + \beta) - \alpha = \beta\,; \quad \alpha - (\beta + \gamma) = (\alpha - \beta) - \gamma\,; \quad \alpha.(\beta - \gamma) = \alpha.\beta - \alpha.\gamma\,.$$

   Show that for any ordinal α there are only finitely many ordinals of the form $\alpha - \beta$.

7. Show that for ordinals α and $\beta \neq 0$ there are unique ordinals γ and δ with $\alpha = \beta.\gamma + \delta$ and $\delta < \beta$.
   Are there always γ, δ with $\alpha = \gamma.\beta + \delta$ and $\delta < \beta$?

8. Show that the synthetic definition of ordinal addition given in lectures is equivalent to the recursive definition

$$\alpha + 0 = \alpha \; ; \quad \alpha + (\beta + 1) = (\alpha + \beta) + 1 \; ; \quad \alpha + \lambda = \sup_{\beta < \lambda} (\alpha + \beta) \; .$$

Show that the synthetic definition of ordinal multiplication given in lectures is equivalent to the recursive definition

$$\alpha.0 = 0 \; ; \quad \alpha.(\beta + 1) = (\alpha.\beta) + \alpha \; ; \quad \alpha.\lambda = \sup_{\beta < \lambda} (\alpha.\beta) \; .$$

9. Show that λ is a limit ordinal if and only if $\lambda = \omega.\gamma$ for some $\gamma \neq 0$.
For what ordinal γ do we have $\epsilon_0 = \omega.\gamma$? For what ordinal γ do we have $\omega_1 = \omega.\gamma$?

10. An ordinal written as $\omega^{\alpha_1}.n_1 + \ldots + \omega^{\alpha_k}.n_k$, where $\alpha_1 > \ldots > \alpha_k$ are ordinals (and k and $n_1, \ldots, n_k$ are non-zero natural numbers), is said to be in *Cantor Normal Form.*
Show that every non-zero ordinal has a unique Cantor Normal Form.
What is the Cantor Normal Form for the ordinal ϵ_0? And for ω_1?

11. Let α be a countable (non-zero) limit ordinal. Prove that there exists an increasing sequence $\alpha_1 < \alpha_2 < \alpha_3 < \ldots$ with supremum equal to α. Is this result true for $\alpha = \omega_1$?

12. Show that, for every countable ordinal α, there is a subset of $\mathbb{Q}$ of order-type α. Why is there no subset of $\mathbb{R}$ of order-type ω_1?

Comments, corrections and queries can be sent to me at `m.hyland@dpmms.cam.ac.uk`.