Set Theory and Logic: Example Sheet 1

1. In lectures we saw subsets of the reals of order types $\omega + \omega$ and ω^2. What were they? Write down a subset of the reals of order type ω^3.

2. Let A be a well-ordering of order type α. Show that the set of initial segments of A is well-ordered by inclusion and determine its order type.

3. What is the least ordinal α such that $1 + \alpha = \alpha$? The next least? The one after that? What is the least ordinal α such that $\omega + \alpha = \alpha$? The next least? The one after that? What is the least ordinal α such that $\omega.\alpha = \alpha$? The next least? The one after that? What about $\alpha.\omega = \alpha$?

4. Prove that $\alpha.(\beta + \gamma) = \alpha.\beta + \alpha.\gamma$ and that $\alpha.(\beta.\gamma) = (\alpha.\beta).\gamma$.
What about $(\alpha + \beta).\gamma = \alpha.\gamma + \beta.\gamma$?

5. Establish the following properties of ordinal subtraction
$$(\alpha + \beta) - \alpha = \beta \, ; \quad \alpha - (\beta + \gamma) = (\alpha - \beta) - \gamma \, ; \quad \alpha.(\beta - \gamma) = \alpha.\beta - \alpha.\gamma \, .$$
Show that for any ordinal α there are only finitely many ordinals of the form $\alpha - \beta$.

6. Show that for ordinals α and $\beta \neq 0$ there are unique ordinals γ and δ with $\alpha = \beta.\gamma + \delta$ and $\delta < \beta$.
Are there always γ, δ with $\alpha = \gamma.\beta + \delta$ and $\delta < \beta$?

7. Suppose that X is a set of ordinals with no maximal element. Show that $\sup X$ cannot be a successor ordinal.

8. Show that the synthetic definition of ordinal addition given in lectures is equivalent to the recursive definition
$$\alpha + 0 = \alpha \, ; \quad \alpha + (\beta + 1) = (\alpha + \beta) + 1 \, ; \quad \alpha + \lambda = \sup_{\beta < \lambda}(\alpha + \beta) \, .$$
Show that the synthetic definition of ordinal multiplication given in lectures is equivalent to the recursive definition
$$\alpha.0 = 0 \, ; \quad \alpha.(\beta + 1) = (\alpha.\beta) + \alpha \, ; \quad \alpha.\lambda = \sup_{\beta < \lambda}(\alpha.\beta) \, .$$

9. Show that λ is a limit ordinal if and only if $\lambda = \omega.\gamma$ for some $\gamma \neq 0$.
For what ordinal γ do we have $\epsilon_0 = \omega.\gamma$? For what ordinal γ do we have $\omega_1 = \omega.\gamma$?

10. An ordinal written as $\omega^{\alpha_1}.n_1 + \ldots + \omega^{\alpha_k}.n_k$, where $\alpha_1 > \ldots > \alpha_k$ are ordinals (and k and $n_1, \ldots, n_k$ are non-zero natural numbers), is said to be in *Cantor Normal Form*.
Show that every non-zero ordinal has a unique Cantor Normal Form.
What is the Cantor Normal Form for the ordinal ϵ_0? And for ω_1?

11. Let α be a countable (non-zero) limit ordinal. Prove that there exists an increasing sequence $\alpha_1 < \alpha_2 < \alpha_3 < \ldots$ with supremum equal to α. Is this result true for $\alpha = \omega_1$?

12. Show that, for every countable ordinal α, there is a subset of $\mathbb{Q}$ of order-type α. Why is there no subset of $\mathbb{R}$ of order-type ω_1?

13. Suppose that A is a well-ordered set. Show that any subset of A is order isomorphic to an initial segment of A.
    Let X be a totally ordered set such that every subset of X is isomorphic to some initial segment of X. Prove that the total ordering is in fact a well-ordering.

14. Suppose that we are given $<$ a (strict) partial order on a set X. Show that $<$ can be extended to a total order on the same set X.

15. In this question do not assume the axiom of choice!
    Suppose that we know only that every set can be totally ordered. Show that any family of finite non-empty subsets of a set has a choice function.

16. (i) Suppose given a well-ordering $<$ of A. We use it to construct a choice function $f : P_{\neq\emptyset}(A) \to A$, by setting $f(x)$ to be the $<$-least element of x. Show that

$$f(x \cup y) = f(\{f(x), f(y)\})\,.$$

    (ii) Suppose now given a choice function $f : P_{\neq\emptyset}(A) \to A$ satisfying the above condition. Show that there is a well-ordering of A which induces this choice function.

Comments, corrections and queries can be sent to me at `m.hyland@dpmms.cam.ac.uk`.