Rome 75   Meaning and the λ-calculus.

What I have to say today falls into three parts.
① A discussion of the views of meaning which lie behind our enquiry, and of how our results reflect on those views.
② Sketches of the methods of proof.
③ Open problems.

① Our (or the historical) position with respect to the λ-calculus is an odd one; for though we can determine the simple rules of the calculus of pure functionality, we are at a loss to understand the meaning of most terms. so it's not surprising ~~Indeed this resulted~~ there was a ~~in the~~ conflict between Church (who thought only normal forms or terms reducible to such meaningful) & Curry (who didn't believe in meaning). Church's view rests on some such belief as that "only fully analyzable terms have meaning"; there is little to be said for this, and anyway however one may regard it for the λI-calculus, it fails completely for λK-calculus.

It's my view that the [illegible] theory of meaning is the intuit. one

If one understands (on an intuitionistic theory of meaning) that f is a function when one can recognize that for anything belonging to the dom(f), the result of application is something which we can recognize (as belonging to rge(f)), then $\lambda x.x$ all right but $\lambda x.xx$ problematical as we "can't tell whether we know what it means or not". So that won't do.

Thus we are left with the possibilities: (a) constr a semantics of some sort; (b) interpret meaning in a way which is new for formal logicians. Let us concentrate on (b) to start with. The clue for (b) comes from the analogy with programming languages.

Analogy:

Terms	States
Reductions	Comp^n steps
Normal forms	Terminated computn

Here we are in the world of practical men,

So the natural suggestion is to understand meaning as use, where use is in computing so reducing away. Of course our notion of use is more restricted than that associated with ~~[illegible]~~ Wittgenstein; for we are only interested in the final results of our use, not in how we get

there; or to put it concretely, we want $(\lambda x.P)Q$ & $P[Q/x]$ to mean the same. [Note that it is only in view of the Church-Rosser Theorem that this is consistent with our proposals].
Since terms may be used as subterms of others (i.e. inside contexts) to give reductions to normal form, the meaning as use proposal amounts to something like this:
M and N have same meaning iff whenever $C[M]$ has n.f. then $C[N]$ has (same) n.f. & ~~conversely~~ vice-versa: so there is implicitly an ordering,
$M < N$ iff whenever $C[M]$ has n.f. then $C[N]$ has (same?) n.f.
In other words, terms meaningful to the extent they contribute to terminating compns.
This brings us to
<u>Discovery</u> (Barendregt, Wadsworth) $\exists$ terms which never make a contribution to termination (i.e. no h.n.f.s or unsolvable terms).
We regard these terms as meaningless & g call an order <u>sensible</u> iff it sets all meaningless terms equal and at the bottom of the meaning order. Thus we can generate a minimal sensible p.o.r. first proved consistent by Barendregt.
Note that the sorts of relns suggested by the meaning is use view are all p.o.r.'s i.e. all satisfy the minimal requirement on a good semantics of preservation under context substitution.

which we consider - we don't gg deal with interpretations ptl rec. fns.

Turn from meaning as use to semantics. There are essentially two types of semantics - one is real semantics about which more later - the other is obtained by extending the Church view in the light of Discovery above.
Introduce a constant Ω to stand for the terms with no h.n.f. Then for any term we have normal forms with Ω's which approximate them. Since the meaningless terms are closed under final abstraction, & the application of meaningless gives meaningless we get Ω-reduction,

$$\Omega M \;\&\; \lambda x.\Omega \quad \geq \quad \Omega$$

& $\beta\Omega$, by Ω-normal forms.

This avoids problems in Lévy Wadsworth's treatments

Then we have ~~relations~~ sets of approxns.
$\omega(M) =$
$\omega\eta(M) =$

and relations

$$w(M) \le w(N)$$
$$w\eta(M) \le w\eta(N).$$

Of course these relations are really syntactic, but they can be presented in a lattice theoretic fashion in the manner of Scott's Lattice of flow diagrams (see Levy, Welch).
The main trouble with these relations is that they do not obviously satisfy the substitution requirement. That they do so, follows from the

<u>Charn Thms</u> (2.3) & (2.6).

$w\eta(M) \le w\eta(N)$ iff $C[M]$ has n.f $\Rightarrow C[N]$ has n.f.
iff $C[M]$ has n.f. $\Rightarrow C[N]$ has $\beta\eta$-equaln.
(ie $C[M] \ge_{\beta\eta}$ n.f. $\Rightarrow C[N] \ge_{\beta\eta} L$).

$w(M) \le w(N)$ iff $C[M] \ge_\beta$ h.n.f. $L \Rightarrow C[N] \ge_\beta L'$ h.n.f similar to L.

<u>Remarks</u> 1) $C[M] \ge_\beta L$ n.f. $\Rightarrow C[N] \ge_\beta L$ doesn't seem to correspond to an approxn ordering.
2) $w(M) \le w(N)$ charn suggests generalizing our notion of use.
3) The relation with which Levy is concerned is not $w(M) \le w(N)$ as stated, but doesn't allow $\lambda x.\Omega \ge \Omega$; it has charn

$A(M) \le A(N)$ iff 1) $C[M] \ge_\beta$ hnf $L \Rightarrow C[N] \ge_\beta L'$ (similar
2) $C[M] \ge_\beta \lambda x_1 \ldots x_k.(\;) \Rightarrow C[N] \ge_\beta \lambda x_1 \ldots x_k.(\;)$.

which shows its a p.o.s (not sensible!).

The proper Semantics are the models $P\omega$ & D_∞ of Scott! These clearly are ideal; the induced order is substitution preserved etc. What is more they have characterizations given by generalized notions of use (i.e. by dropping the obsession with computations which converge).

<u>Charn Thms</u> (3.2)

$[\![M]\!]_{P\omega} \le [\![N]\!]_{P\omega}$ iff $C[M] \ge_\beta$ hn.f. $L \Rightarrow C[N] \ge_\beta$ hnf. L' which is more functional (but inseparable) L

$[\![M]\!]_{D_\infty} \le [\![N]\!]_{D_\infty}$ iff $C[M]$ has hn.f $\Rightarrow C[N]$ has hnf.
iff $C[M]$ has hnf $\Rightarrow C[N]$ has inseparable hnf.

<u>Remarks</u> 1) The last is miraculous: it is chard by a) knowing what is meaningless & b) $M < N$ iff whenever $C[M]$ has meaning then so has $C[N]$:
further it does the beautiful
unique maximal consistent sensible p.o.s.
2) Finally let me emphasize that it is meaning as use which runs as a thread thro' all this

② The chief tool in establishing the connections between the lattice models and the meaning as use chars, ~~guma~~ is essentially to be found in the proof of Bohm's Theorem.

<u>Defn</u>: h.n.f.'s <u>insepble</u> iff when we have the as $\lambda x_1 \ldots x_m . z X_1 \ldots X_i$, $\lambda x_1 \ldots x_n . w Y_1 \ldots Y_j$, the z is w & $m-i = n-j$.

<u>Main point</u> (1.2) If M, N have h.n.f.s which aren't insep. then M, N are incompble in any consis p.o.r. which sats. β-equality.

If have insepble h.n.f.'s then apply to $x_1 \ldots x_m$ ($m \geq n$) & get $z X_1 \ldots X_i$, $w Y_1 \ldots Y_j$ & can now set about comparing (X_r, Y_r) [Nakajima essentially involved with a special case of this]. Then, can repeat process.

<u>Main point</u> (1.3) If we come on not insepble corresponding bits of M, N, then can apply a context to pick them out (apart from some trivial substitutions which don't change insepbltyet

Bohm's Theorem an immediate corollary.

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Now a couple of sketch proofs.

(2.3) $w(M) \subseteq w(N)$ iff $C[M]$ has$_\beta$ hnf $\Rightarrow C[N]$ has$_\beta$ similar hnf

(<u>Remark</u> RHS makes sense by (1.1).)

Proof :- ($\Rightarrow$) Immediate from (2.1)(b) result of a sort due to Wadsworth.

($\Leftarrow$) <u>Lemma 2.2</u> if $L \notin w(N)$, then $\exists$ k-pairs of (L, N), (X, Y) with X has hnf but Y has not similar hnf.

Now if $L \in w(M) \setminus w(N)$ we get $C[\;]$ s.t. $C[L]$ has hnf but $C[N]$ has not similar hnf, by Bohm type analysis & hence not RHS by Wadsworth result (easy part).

<u>Note</u>: This easily proves char of $A(M) \subseteq A(N)$.

The char of Morris reln is more tricky.