<u>Cartesian closed co-reflective subcategories of TOP.</u>

(Some definitions and results used here are from my "Topological spaces, limit spaces and continuous lattices" hereafter referred to as TS).

Let $S \subseteq$ TOP (or any other category which is suitably co-complete); define $\sigma : \text{TOP} \longrightarrow \text{TOP}$ by
$$\sigma(X,\tau) = \begin{cases} X \text{ with the finest topology} \\ \text{such that all maps from element} \\ \text{of } S \text{ to } X \text{ stay continuous} \end{cases}$$
(i.e. the colimit of the diagram which consists of all maps from elements of S to X).
Then $\text{corh}(S)$, the <u>coreflective hull of S</u> in TOP is $\sigma(\text{TOP}) = \{(X,\tau) \mid \sigma((X,\tau)) = (X,\tau)\}$.
$\text{corh}(S)$ is the least coreflective subcategory of TOP ~~[illegible]~~ containing S, and S is <u>dense</u> in $\text{corh}(S)$ (i.e. every element of $\text{corh}(S)$ is a canonical colimit of elements of S).

The main aim of this note, is to give a simple proof of the following,

<u>Theorem 1</u>. If $S \subseteq$ TOP is closed under product and consists of elements which have function spaces (see TS), then $\text{corh}(S)$ is cartesian closed.

<u>Remark</u>. This is sometimes stated in an apparently stronger form namely "If $S \subseteq$ TOP consists of elements which have function spaces, and the product of elements of S is in $\text{corh}(S)$, then $\text{corh}(S)$ is cartesian closed."
This added generality is illusory. Let S_1 consist of the finite products of elements of S. Clearly all the elements of S_1 have function spaces ($X^{Y\times Z}$ is $(X^Y)^Z$). Also note that if $A \in S$ and $B \in \text{corh}(S)$, then $A \times B \in \text{corh}(S)$ (we have $A\times B = A \times \text{coli}(B_\alpha) = \text{coli}(A\times B_\alpha)$ – but $A \times B_\alpha \in \text{corh}(S)$). Thus all elements of S_1 are in $\text{corh}(S)$. Hence $\text{corh}(S) = \text{corh}(S_1)$ and Theorem 1 can be applied to S_1.

In our proof of Theorem 1, we make use of the reflection of TOP in the category LIM of limit spaces. But to make the proofs as simple as possible we drop the requirement that the intersection of two filters converging to a point should also converge to that point. (The two trivial axioms ensure that LIM is cartesian closed).

Now given S as in Theorem 1, we wish to consider its image $L(S)$ in LIM. We adopt the convention of priming everything to indicate that we are in LIM not TOP. $L(S)$ is S' and we have a diagram as follows:

$$\begin{tikzcd}
\mathrm{coh}(S') \arrow[r, shift right, "i'"'] & \mathrm{LIM} \arrow[l, shift right, "\sigma'"'] \arrow[d, shift right, "T"'] \\
\mathrm{coh}(S) \arrow[r, shift right, "i"'] & \mathrm{TOP} \arrow[l, shift right, "\sigma"'] \arrow[u, shift right, "L"']
\end{tikzcd}$$

We easily see that,

<u>Lemma 2</u>. i' preserves products.

Proof:- Let (X, Λ), (Y, M) be in $\mathrm{coh}(S')$. A product in $\mathrm{coh}(S')$ is obtained by applying σ' to an ordinary product in LIM. So it is sufficient to show that if the filter $\Theta \downarrow (x,y)$, then for some $C \in S'$ $f : C \to (X \times Y, \Lambda \times M)$ $\Phi \downarrow c$ in C with $f(c) = (x,y)$, we have $\Theta \geq f(\Phi)$. Let Θ_X, Θ_Y denote the projections of Θ on X, Y respectively. There is C_X, C_Y ; $f_X, f_Y : C_X, C_Y \to X, Y$; $\Phi_X \downarrow c_X$, $\Phi_Y \downarrow c_Y$; $f_X(c_X) = x$, $f_Y(c_Y) = y$; with $\Theta_X \geq f_X(\Phi_X)$, $\Theta_Y \geq f_Y(\Phi_Y)$. Set $C = C_X \times C_Y$, $f = (f_X \times f_Y)$ etc and we are home.

From Lemma 2, we easily deduce the "primed version" of Theorem 1.

<u>Lemma 3</u>. $\mathrm{coh}(S')$ is cartesian closed.

Proof:- Let A, B, C be in $\mathrm{coh}(S')$.

$$\begin{aligned}
\mathrm{Mor}_{\mathrm{coh}(S')}(C, \sigma'([i'(B), i'(A)])) &\cong \mathrm{Mor}_{\mathrm{LIM}}(i'C, [i'B, i'A]) \\
&\cong \mathrm{Mor}_{\mathrm{LIM}}(i'C \times i'B, i'A) \\
&\cong \mathrm{Mor}_{\mathrm{LIM}}(i'(C \times B), A) \\
&\cong \mathrm{Mor}_{\mathrm{coh}(S')}(C \times B, A).
\end{aligned}$$

So $\sigma'([i'(B), i'(A)])$ is function space of B to A in $\mathrm{coh}(S')$.

Having obtained what we want in the primed version, it remains to pull it down. So we must connect $\mathrm{coh}(S')$ with $\mathrm{coh}(S)$. There is a fairly obvious way to do this. First, since any element of $\mathrm{coh}(S)$ is a canonical colimit of elements of S, we can map it to the corresponding colimit in $\mathrm{coh}(S')$ (-this colimit will also be canonical). This gives $L_C : \mathrm{coh}(S) \to \mathrm{coh}(S')$ where L_1 is $\sigma' \circ L \circ i$. Secondly, we can map a canonical colimit in $\mathrm{coh}(S')$ to the corresponding (not necessarily canonical) colimit in corh(S). This gives $T_C : \mathrm{corh}(S') \to \mathrm{corh}(S)$, where T_C is $\sigma_0 T_0 i'$ and $i_0 T_C = T_0 i'$, clearly.

Lemma 4. $L_C(\mathrm{corh}(S))$ is a reflective subcategory of $\mathrm{corh}(S')$; (L_C is full and faithful).

Proof:- $$\mathrm{Mor}(T_C(X,\Lambda),(Y,\tau)) \cong \mathrm{Mor}(i_0 T_C(X,\Lambda), i(Y,\tau))$$
$$\cong \mathrm{Mor}(T_0 i'(X,\Lambda), i(Y,\tau))$$
$$\cong \mathrm{Mor}(i'(X,\Lambda), L_0 i(Y,\tau))$$
$$\cong \mathrm{Mor}((X,\Lambda), \sigma'_0 L_0 i(Y,\tau)).$$ #

In view of this lemma we identify corh(S) with $L_1(\mathrm{corh}(S))$ - as we do with TOP and LIM.

Lemma 5. The reflector $T_C : \mathrm{corh}(S') \to \mathrm{corh}(S)$ preserves products.

Proof:- Let $A, B \in \mathrm{corh}(S')$, $A = \mathrm{colim}(A_\alpha)$, $B = \mathrm{colim}(B_\beta)$.

$T_C(A \times B) = T_C(\mathrm{colim}(A_\alpha) \times \mathrm{colim}(B_\beta))$

$= T_C(\mathrm{colim}\ A_\alpha \times B_\beta)$ since by Lemma 3 $\times$ is a left adjoint in corh(S'),

$= \mathrm{colim}\ T_C(A_\alpha \times B_\beta)$ since T_C is left adjoint,

$= \mathrm{colim}(A_\alpha \times B_\beta)$ as $A_\alpha \times B_\beta \in S$

$= \sigma\ \mathrm{colim}_{TOP}(A_\alpha \times B_\beta)$

$= \sigma(\mathrm{colim}_{TOP} A_\alpha \times \mathrm{colim}_{TOP} B_\beta)$ as $A_\alpha \times$, $\times B_\beta$ are both left adjoints in TOP as they are in S.

$= T_C A \times T_C B$.

Lemma 6. If a reflector $R : \underline{C} \to \underline{D}$ preserves products and $\underline{C}$ is cartesian closed, then so is $\underline{D}$ (with cartesian closed structure mapped down by ψ).

Proof:- If D, D_1, D_2 are in $\underline{D}$, "[,]" is function space in $\underline{C}$
$$\mathrm{Mor}_{\underline{D}}(D, \psi[D_1,D_2]) \cong \mathrm{Mor}_{\underline{C}}(D,[D_1,D_2]) \cong \mathrm{Mor}_{\underline{C}}(D \times_C D_1, D_2)$$
$$\cong \mathrm{Mor}_{\underline{D}}(D \times_D D_1, D_2).$$

Theorem 1 is now easily obtained by applying Lemmas 5 and 6 to Lemma 4. Our lemmas also readily provide some further information.

Proposition 7 Let S_1, S_2 be as S in Theorem 1, and let $\mathrm{corh}(S_1) \subseteq \mathrm{corh}(S_2)$. Then the inclusion preserves products and the cartesian closed structure on $\mathrm{corh}(S_1)$ is induced by the coreflector from that on $\mathrm{corh}(S_2)$.

Proof:- w.l.g. we may assume $S_1 \subseteq S_2$ (cf the remark following the statement of Theorem 1). Then $\mathrm{corh}(S_1') \subseteq \mathrm{corh}(S_2')$; call this inclusion I' & the one we want I. Then $I = T_{C_2} \circ I' \circ L_{C_1}$ (because clearly $T_{C_1}(X)$ may have less filters converging then $T_{C_2}(X)$, but they must have the same induced topology). L_{C_1} is right adjoint so preserves products; T_{C_2} does so by Lemma 5; I' is $\sigma_2' \circ i_1'$ and σ_2' is right adjoint preserves products, while i' does so by Lemma 2. To see that if $A, B \in \mathrm{corh}(S_1)$ then $[A,B]_{\mathrm{corh}(S_1)}$ is the coreflection of $[A,B]_{\mathrm{corh}(S_2)}$ is easy.

<u>Proposition 8</u>. If S is as in Theorem 1, $A, B \in \mathrm{corh}(S)$ and $[A,B]$ a function space in TOP, then $\sigma([A,B])$ is the function space in $\mathrm{corh}(S)$.

Proof:- It is sufficient to check that $\sigma([A,B])$ and $[A,B]_{\mathrm{corh}(S)}$ admit the same maps into them from elements of S. But if $C \in S$, $A \in \mathrm{corh}(S)$, $C \times_{\mathrm{corh}(S)} A$ is $C \times A$; hence

$$\mathrm{Mor}(C, [A,B]_{\mathrm{corh}(S)}) \cong \mathrm{Mor}(C \times_{\mathrm{corh}(S)} A, B) \cong \mathrm{Mor}_{\mathrm{TOP}}(C, [A,B]) \cong$$
$$\cong \mathrm{Mor}_{\mathrm{corh}(S)}(C, \sigma[A,B]).$$

We note that what we have just used to prove Prop. 8 (and used in Lemma 5) generalizes to,

<u>Proposition 9</u>. If at least one of $A, B \in \mathrm{corh}(S)$ has function spaces in TOP, then $A \times_{\mathrm{corh}(S)} B$ is $A \times B$.

Proof:- Let A have function spaces and $B = \mathrm{colim}\, B_\alpha$.

$$A \times B = A \times \mathrm{colim}\, B_\alpha = \mathrm{colim}\, A \times B_\alpha = \mathrm{colim}\, A \times_{\mathrm{corh}(S)} B_\alpha$$
$$= A \times_{\mathrm{corh}(S)} B.$$

<u>Remark</u> This goes some way to explaining why it is hard to find counter-examples to "i preserves products". For example, amongst Hausdorff sequential spaces, if $X \times Y$ in TOP is to be non-sequential for sequential X, Y, then

i) one of X, Y must be non-1st-countable

ii) neither X, Y may be locally compact.

In fact, $X = [\mathbb{N}, \mathbb{N}]^{\smile}$, $Y = [X, \mathbb{N}]$ will do.

There is a maximal coreflective subcategory of TOP, of the sort we are considering. Take for S in Theorem 1, the collection of all topological spaces which have function spaces. For this S, $\mathrm{corh}(S)$ strictly includes $\mathrm{corh}(K)$ where K is the collection of compact Hausdorff spaces.