A Dialectica-style Interpretation of Type Theory

Symposium in honour of Pierre-Louis Curien

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For Pierre-Louis Curien at 60:

A talk on the semantics of Type Theory being a reflection of his broad interests

Programming Languages and Proof Theory

- Logic, Category Theory and Theoretical Computer Science
- Foundations of Type Theory
  P.-L. Curien, R. Garner and M. Hofmann. *Revisiting the categorical interpretation of type theory*

An Excursion into Categorical Proof Theory

Realizability and Functional Interpretations

- Traditional Proof Theory: Indexed Preorders
- Categorical Proof Theory: Indexed Categories

Many (most) traditional interpretations are the preordered set reflection of a natural categorical proof theory.

60th Birthday Question

Raised by Per Martin-Löf at the Commemorative Symposium dedicated to Anne S. Troelstra on the Occasion of His 60th Birthday: September 1999, Noordwijkerhout, The Netherlands

Is there a Dialectica Interpretation of Type Theory?

Motivation

A judgement

$$t(a_1, \cdots a_n) \in B \quad [a_1 \in A_1. \cdots a_n \in A_n]$$

already has the shape of the Dialectica Interpretation. So prima facie Type Theory renders the interpetation redundant.

Summary and Outline

Dialectica-style Interpretations

A number of interpretations use the Dialectica idea. A good way to approach them is via an unexpected interpretation of Type Theory based on polynomials or containers.

Contents

- Background on the Dialectica interpretation
- Type Theory and categorical models.
- Polynomials or containers as fibred categories
- von Glehn fibrations: the polynomial interpretation
- Concluding remarks on the Dialectica interpretation

The Dialectica Interpretation

Roots in mathematical logic

- K. Gödel. *Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes*. Dialectica, 1958. Interpretation of Heyting arithmetic in primitive recursives functionals of finite typesystem T via formulae

$$\exists u.\forall x.A(u,x)$$

- J.-Y. Girard. Interprétation fonctionelle et élimination des coupures de l'arithmetique d'ordre superieure, Paris VII, 1972.
  Second order system F and extension of the interpretation.
- A. S. Troelstra. *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*. SLN 334, 1974.
- U. Kohlenbach. Monotone interpretation: proof mining. 1990-

The Dialectica Interpretation

Dialectica Categories

de Paiva, 1986: The Dialectica implication as maps in a category:

- objects $U \leftarrow A \rightarrow X$
- maps $U \leftarrow A \rightarrow X$ to $V \leftarrow B \rightarrow Y$
  - $f : U \rightarrow V$
  - $F : U \times Y \rightarrow X$
  - $\phi : \Pi u \in U y \in Y . A(u, F(u, y)) \rightarrow B(f(u), y)$

Originally $U \leftarrow A \rightarrow X$ was a relation between U and X and so ϕ an inclusion.

Variants

- Girard Categories and Linear Logic.
- Diller-Nahm monad: cartesian closed categories

Folklore Understanding of the Dialectica

Read $U \leftarrow A \rightarrow X$ as $\Sigma u \in U . \Pi x \in X . A$. The Dialectica maps reflect the idea that the Σ and Π have been added freely.

Related ideas

- Simple games as free Πs of free Σs of free Πs. (Various discussions by Cockett, Seeley and others.)
- A. Joyal. *Free bicompletion of enriched categories* and *Free bicomplete categories.* C. R. M. Rep. Acad. Sci. Canada, 1995. (Information on the additives of Linear Linear Logic.)
- P. Hofstra. *The dialectica monad and its cousins.* CRM Proceedings and Lecture Notes 53. (Precise analysis for simple products and sums.)

Free sums

Ingredients: $\mathbb{E} \to \mathbb{B}$ a fibration assumed cloven; $\mathcal{F}$ a suitable class of maps in $\mathbb{B}$.

Output: Category $\Sigma\mathbb{E} = \Sigma_{\mathcal{F}}(\mathbb{E} \to \mathbb{B})$. Concretely we have

Objects Over $b \in \mathbb{B}$: $x \in \mathbb{E}(b')$ with $u : b' \to b$.

Maps From y with $v : c' \to c$ to x with $u : b' \to b$:

$$\begin{array}{ccc} c' & \xrightarrow{v} & c \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ b' & \xrightarrow[u]{} & b \end{array} \qquad\qquad y \to f'^{*}(x)$$

over $f : c \to b$.

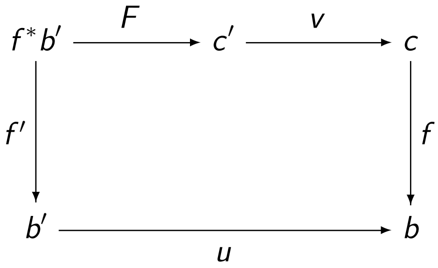
Free products

Ingredients: $\mathbb{E} \to \mathbb{B}$ a fibration assumed cloven; $\mathcal{F}$ a suitable class of maps in $\mathbb{B}$.

Output: Category $\Pi\mathbb{E} = \Pi_{\mathcal{F}}(\mathbb{E} \to \mathbb{B}) = \big(\Sigma(\mathbb{E}^{op})\big)^{op}$.
Concretely we have

Objects Over $b \in \mathbb{B}$: $x \in \mathbb{E}(b')$ with $u : b' \to b$.

Maps From y with $v : c' \to c$ to x with $u : b' \to b$:



$$F^* y \to f'^*(x)$$

Dependent Type Theory

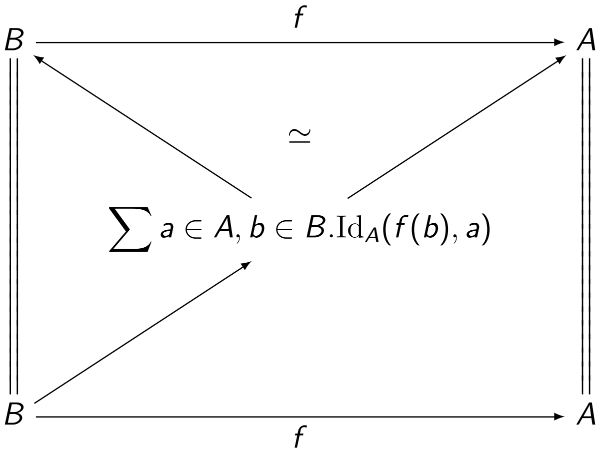
Unrealised intention: A General theory of inductive definitions

Main Ingredients

- Typed terms: $a \in A \vdash t(a) \in B$
- Types indexed over types: $a \in A \vdash B(a)\ \mathrm{type}$
- Implicit (!) Substitution: From $a \in A \vdash t(a) \in B$ and $b \in \vdash C(b)\ \mathrm{type}$ get $a \in C(t(a))\ \mathrm{type}$.
- Sums and products: If $a \in A \vdash B(a)\ \mathrm{type}$ then $\Sigma a \in A.B(a)\ \mathrm{type}$ and $\Pi a \in A.B(a)\ \mathrm{type}$ with familiar rules (left and right adjoints with Beck-Chevalley)
- Identity types $a, a' \in A \vdash \mathrm{Id}_A(a, a')\ \mathrm{type}$

Identity and factorisation

The interaction of identity and existence



The Sceptics Case

Why a Dialectica Interpretation of Type Theory seems problematic

The Type Theoretic Axiom of Choice

$$\Pi a \in A.\Sigma b \in B(a).C(b) \rightarrow$$

$$\Sigma f \in (\Pi a \in AB(a)).\Pi a \in A.C(f(a))$$

In typical circumstances the map is an isomorphism.

Consequence of AC

Inductively all types have the form

$$\Sigma u \in U.\Pi x \in X(u).A(x)$$

which is already the form of the Dialectica interpretation.

Categorical Models

Categories with Fibrations or Display Maps

Category $\mathbb{C}$ with collection of maps $\mathbb{F}$ in $\mathbb{C}$.

- $\mathbb{F}$ closed under pullback so we form a fibration $\mathbb{F} \rightarrow \mathbb{C}$
- $\mathbb{F}$ contains all isos and is closed under composition
- Pullbacks along maps in $\mathbb{F}$ have right adjoints
- Every map factorizes as a trivial cofibration followed by a fibration (Trivial cofibration = llp wrt fibrations.)

Additional assumption

- Each slice $\mathbb{F}/I$ has finite coproducts and these are stable
- All maps to 1 are in $\mathbb{F}$

Remarks

For today call a $(\mathbb{F} \to \mathbb{C})$ satisfying the assumptions a category with fibrations.

- Closure under composition gives
  - gives strong sums
  - ensures Beck-Chevalley = good substitution
- The right adjoints give products
- The factorization ensures identity types

Terminological problem

What to about the clash?

- $\mathbb{F} \to \mathbb{C}$ is a categorical fibration
- The arrows in $\mathbb{F}$ are called fibrations by reason of a topological intuition. Display map is a bit technical.

Constructing Interpretations of Dependent Types

New models from old: the simple case

Restriction from locally cartesian closed categories

Suppose we have a construction taking locally cartesian closed categories to locally cartesian closed categories. We try to refine it to a direct construction on interpretations of Type Theory e.g. our categories of fibrations.

Examples

- Presheaf and sheaf models
- Realizability and its variants

Taking the Fibration seriously

We have a construction on categories which we apply fibrewise to $\mathbb{F} \to \mathbb{C}$ giving $\Phi(\mathbb{F}) \to \mathbb{C}$ keeping the base fixed. We need to extend this fibration along a functor

$$\mathbb{C} = \mathbb{F}(1) \longrightarrow \Phi\mathbb{F}(1)$$

to give a model of Type Theory. This is not straightforward.

One possibility is *internalisation*, the technique for the Dialectica Interpretation of System F in Girard's Thesis. Another lies behind this talk. There is at least one more.

Polynomials

Also known as Containers

Polynomials and maps of polynomials in **Sets**

A polynomial is a map $U \leftarrow X$thought of as a general signature: a collection U of function symbols with X_u the arity of u. A map of polynomials from $U \leftarrow X$ to $V \rightarrow Y$ is

$$\begin{array}{ccccc} U & \longleftarrow & X & \stackrel{F}{\longleftarrow} & f^*Y \\ {\scriptstyle f}\downarrow & & & & \downarrow \\ V & & \longleftarrow & & Y \end{array}$$

This gives a category **Pol** of polynomials or in a different culture containers.

The fibred category of polynomials

There is an evident fibred version of the polynomial construction.

We can identify that with

$$\Sigma(\mathbf{Sets}^2 \to \mathbf{Sets})^{op} ,$$

the result of freely adding sums to the *opposite* of **Sets** indexed over **Sets**. So **Pol** is the fibre over 1.

Note the use of the opposite of a fibred category!

So for a fibration $\mathbb{E} \to \mathbb{B}$ we define

$$\mathrm{Pol}(\mathbb{E}) = \Sigma(\mathbb{E}^{op})$$

the polynomial construction.

Composition of Polynomials

The bicategory

A polynomial $U \leftarrow X$ induces a functor

$$\mathbf{Sets} \rightarrow \mathbf{Sets} : S \mapsto \sum u \in U.(X_u \Rightarrow S)$$

In a more general perspective indexed polynomials

$$I \leftarrow U \leftarrow X \rightarrow J$$

are the 1-cells of a bicategory with polynomial maps as 2-cells.

Polynomial operads

Monads in the polynomial bicategory correspond exactly to rigid operads, equivalently (Zawadowski) to the operads with non-standard amalgamation of Hermida-Makkai-Power.

Cartesian closure

A Little Miracle

T. Altenkirch, P. Levy and S. Staton. *Higher Order Containers*. In CiE'10, LNCS 6158.

Theorem

The category **Poly** *of containers/polynomials is cartesian closed. But it is not locally cartesian closed.*

Further analysis

P. Hyvernat. *A linear category of polynomial diagrams*. To appear in Mathematical Structures in Computer Science.

- Linear logic background.
- Direct formulation of ALS in type theory. (Removed from final version!) So ALS in any *suitable interpretation*, (e.g. locally cartesian closed categories with coproducts.)

Computing with coproducts

Concrete explanation of Altenkirch-Levy-Staton

Take a coproduct $A + B$. Write it as

$$A + B = \Sigma x \in A + 1. \star(x) \Rightarrow B$$

where $\star(x)$ is represented by the obvious $1 \to A + 1$.
So by (AC) we have

$$\begin{aligned} C \Rightarrow A + B &= C \Rightarrow \big(\Sigma x \in A + 1. \star(x) \Rightarrow B\big) \\ &= \Sigma f \in (C \Rightarrow A + 1).\Pi c \in C.(\star(f(c)) \Rightarrow B) \end{aligned}$$

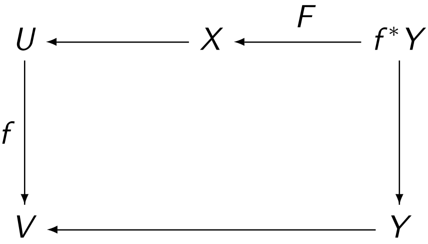
This is the key idea also in the interpretation of Type Theory.

Natural Question on Polynomials

Is there a (simple) notion of fibration for polynomials giving a model of type theory?

Answer of Tamara von Glehn: YES!

Take fibrations to be the maps $U \leftarrow X$ to $V \leftarrow Y$



where $F : f^*Y \to X$ is a coproduct inclusion.

The Polynomial Interpretation

Given $\mathbb{F} \to, \mathbb{C}$ a category with fibrations, the polynomial category with fibrations has

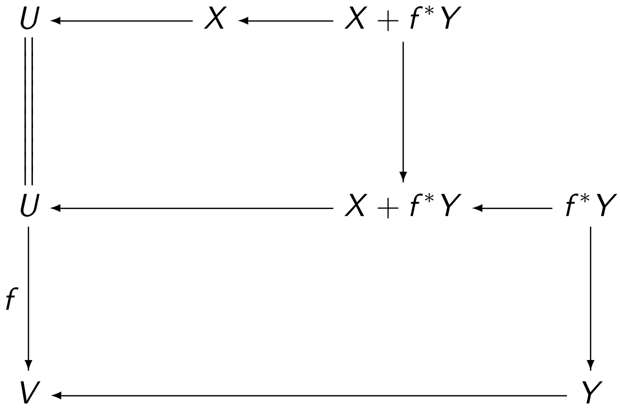
- objects the fibrations $U \leftarrow X$ from $\mathbb{F}$;
- maps the standard maps of polynomials;
- fibrations the von Glehn fibrations.

Theorem

Suppose $\mathbb{F} \to \mathbb{C}$ satisfies the fundamental assumptions. Then so does the corresponding polynomial category with fibrations.

The von Glehn Factorization

For the locally cartesian closed case



Functional extensionality

A calculation in the model

For types A and $B(a)$ $[a \in A]$ consider the identity type on $\Pi a \in A.B(a)$. For $f, g \in \left(\Pi a \in A.B(a)\right)$ function extensionality is

$$(\Pi a \in A.\mathrm{Id}_{B(a)}(f(a), g(a)) \Rightarrow \mathrm{Id}_{\Pi a \in A.B(a)}(f, g)$$

A more delicate question is whether the types either side of $\Rightarrow$ are isomorphic. von Glehn has shown the following.

Theorem

The axiom of extensionality is preserved by the polynomial interpretation, but the isomorphism fails in polynomial models.

The Dialectica Interpretation of Type Theory

Suppose we have $\mathbb{F} \to \mathbb{C}$ a category with fibrations. The Dialectica fibration is by definition $\Sigma\Pi\mathbb{F}$.

Theorem

The Dialectica fibration extends to a category with fibrations

- Either by adapting the von Glehn analysis.
- Or since

$$\mathrm{Pol}(\mathrm{Pol}\mathbb{F}) = \Sigma(\Sigma\mathbb{F}^{op})^{op} = \Sigma\Pi\mathbb{F}$$

  and so one can find it inside the iterated polynomial model.

Aim of the talk

To reflect Pierre-Louis' intense interest in scientific questions and his infectious pleasure in elegant answers

Many Happy Returns Pierre-Louis