

SOME REASONS FOR  
GENERALIZING  
DOMAIN THEORY

MARTIN HYLAND

DOMAINS IX

SUSSEX

SEPTEMBER 2008

# CONTEXT

G. Kreisel. Some reasons for  
generalizing recursion theory

In Proceedings of the Summer  
School and Colloquium in  
Mathematical Logic,  
Manchester 1969

(Editors: Gandy, Yates)

Published 1971

# AIMS

as discussed by Kreisel

(a) Advancing other parts of logic

Finiteness

Definability (Admissible sets)

(b) Understanding the mathematical character of ordinary recursion theory

Axiomatics (Wagner-Strong)

(c) Analysis of the general concept of computation

Church's Thesis Turing

(d) Other uses:

- formal theory of self application (Scott's Do)

- axiomatic analysis of kinds of evidence ?

(Categorical logic: Stone Duality?)

# AIMS

## re Domain Theory

- (a) Advancing other parts of computer science
- (b) Understanding the mathematical character of Scott's Domain Theory
- (c) Analysis of a general concept of approximation
- (d) Other uses

# GENERALIZATIONS

## of Domain Theory

Extensions of the category (SFP)

Other styles of domains (stable)

But for today

Domains as categories

Early work

Taylor

Lamarche

Hyland - Pitts

(At the meeting I was reminded of Lehmann, perhaps the earliest.)

Recent work

Cattani - Winskel

Fiore - Gambino -

Hyland - Winskel

# LINEAR LOGIC

as potential distraction

Cirard translation

$$\begin{aligned}\mathbb{C}(X \times Y, Z) &= \mathbb{L}(! (X \times Y), Z) \\ &= \mathbb{L}(! X \otimes ! Y, Z) \\ &= \mathbb{L}(! X, ! Y \multimap Z) \\ &= \mathbb{C}(X, ! Y \multimap Z)\end{aligned}$$

While (by modified realizability) any cartesian closed category can be obtained from a model of classical linear logic thus,

this is NOT the reason why

Algebraic Domains are  
Cartesian Closed

# RELATIONAL MODEL

for Linear Logic

Rel is compact closed and  
multisets  $M$  gives an exponential  
comonad

Rel together with the cartesian  
closed  $Kl(M)$  (Kleisli category)  
is very rich: it models

differential lambda calculus

fixed points

recursive domain equations

BUT it isn't domain theory

# EXPLANATIONS

$Rel$  is the Kleisli category  $Kl(P)$

where

$$P(X) = (X \Rightarrow \Omega)$$

is the power set monad.

So compact closed and self dual

$M$  multisets is the monad for commutative monoids on  $Sets$  and this extends to  $Rel$  WHY?

Duality turns the monad to a comonad.



# DISTRIBUTIVITY

$M$  and  $P$  monads on  $S$

TFAE

① 
$$\begin{array}{ccc} M\text{-Alg} & \hookrightarrow & \tilde{P} \\ \downarrow & & \\ S & \hookrightarrow & P \end{array}$$
  $P$  lifts to  $M\text{-Alg}$

② 
$$MP \xrightarrow{\lambda} PM$$
  
distributive law

③ 
$$\begin{array}{ccc} KL(P) & \hookrightarrow & \tilde{M} \\ \uparrow & & \\ S & \hookrightarrow & M \end{array}$$
  $M$  extends to  $KL(P)$

and then  $PM$  a monad  $KL(PM) \cong KL(\tilde{M})$

(Analogous results for comonad/monad also important in CS.)

# OPERADS

(linear monads)

For any cocomplete symmetric category  $\mathcal{C}$ ,  
 $\mathcal{C}\text{Mon}(\mathcal{C}) \rightarrow \mathcal{C}$  is monadic, i.e. the  
free commutative monoid monad  $M$   
exists.

The power set  $P$  is commutative  
(it is  $V$ -lattices) so monoidal and  
so lifts to  $\mathcal{C}\text{Mon}(\mathcal{C}) = M\text{-Alg}$

$\therefore M$  extends to  $\text{Kl}(P)$

Same with  $M$  replaced by any  
operadic = linear monad.

Folklore? Eg. Eilenberg-Wright

# PRO FUNCTORS

(Winkler's approach to  
concurrency)

$P(A) = [A^{\text{op}}, \text{Set}]$  category of processes  
of shape.

$$\begin{aligned} \text{Prof}(A, B) &= [A \times B^{\text{op}}, \text{Set}] \\ &= [A, P(B)] \end{aligned}$$

category of set-valued relations  
from  $A$  to  $B$ .

Such a  $A \xrightarrow{F} B$  induces (action on points)

$$P(A) = [A^{\text{op}}, \text{Set}] \longrightarrow [B^{\text{op}}, \text{Set}] = P(B)$$

linear map on processes

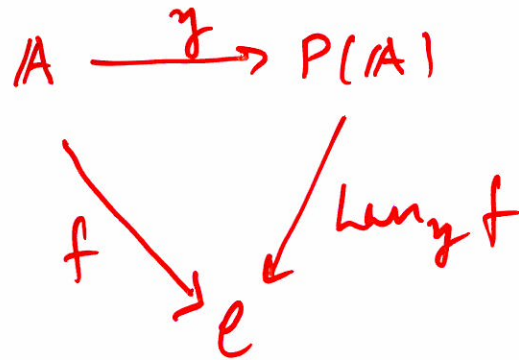
Non-linear maps from (pseudo) comonads  
which arise by extending monads on  
 $\text{Cat}$  to  $\text{KL}(P) = \text{Prof}$

# LIFTS/EXTENSIONS

Fix  $PA = [A^{\text{op}}, \text{Set}]$  (not quite a monad but never mind!) and  $M$  a 2-monad on  $\text{Cat}$



↑  
 This means good behaviour in



for  $M$ -algebras.

# EXAMPLES

of suitable monads

M

terminal object

products

finite limits

(symmetric) monoidal

Resulting comonad adds finite colimits - superficially analogous to domain theory

Models differential  $\lambda$ -calculus w/ fixed points: Joyal's species are in  $Kl(\tilde{M})(1,1)$

# IND-COMPLETION

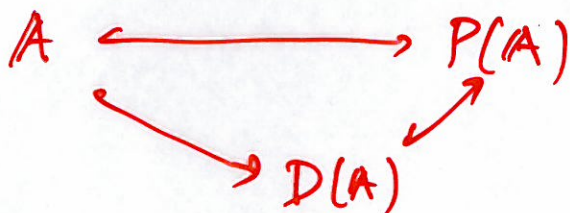
For  $\mathcal{A}$  small we have the Yoneda embedding  $y: \mathcal{A} \longrightarrow P(\mathcal{A}) = [\mathcal{A}^{\text{op}}, \text{Set}]$

Every  $X \in P(\mathcal{A})$  is the colimit over its category of elements  $\mathbb{E}(X)$  of representables.

Let  $D(\mathcal{A})$  consist of the  $X$  with  $\mathbb{E}(X)$  filtered.

$P(\mathcal{A})$  closure of  $\mathcal{A}$  under colimits

$D(\mathcal{A})$  closure of  $\mathcal{A}$  under filtered colimits



# FINITE COLIMITS

If  $\mathcal{C}$  small with finite colimits  
then  $D(\mathcal{C})$  has finite colimits  
(so is cocomplete) and

$$\mathcal{C} \longrightarrow D(\mathcal{C})$$

preserves finite colimits.

Also in

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & D(\mathcal{C}) \\ & \searrow f & \swarrow \hat{f} \\ & \mathcal{X} & \end{array}$$

if  $f$  preserves finite colimits then  
so does  $\hat{f}$  (so it is cocontinuous)

# LIFTS/EXTENSIONS for the Ind Completion

We know

$D$  lifts to  $S\text{-Alg}$

$$\begin{array}{c} S\text{-Alg} \hookrightarrow \tilde{D} \\ \downarrow \\ \text{Cat} \hookrightarrow D \end{array}$$

Hence

$S$  extends to  $KL(D)$

$$\begin{array}{c} KL(D) \hookrightarrow \tilde{S} \\ \uparrow \\ \text{Cat} \hookrightarrow S \end{array}$$

Moreover  $DS$  a monad  $\cong P$

and  $KL(\tilde{S}) \cong KL(DS) \cong KL(P) = \text{Prof}$

Generalized domain theory can  
be extracted from this setting.



## POSET CASE

$D$  monad for directed completion

$S$  monad for  $v$ -lattices

$DS = P$  the "power set" monad

$$A \longmapsto (A^{\text{op}} \Rightarrow 2)$$

on posets

Then

$D$  lifts to  $v$ -lattices =  $S\text{-Alg}$

$S$  extends to  $KL(D)$  category

of (directed relations or) free

depos and Scott continuous maps

# DCPOs

This is the category  $D\text{-Alg}$

$D$  is a (necessarily) commutative monad with

$$D(A \times B) \xrightarrow{\cong} D(A) \times D(B)$$

the inverse of the monoidal structure.

Thus  $D\text{-Alg}$  is cartesian closed.

# ALGEBRAIC LATTICES

Objects are easy  $KL(\hat{D})$  from

the lift  $Alg(S) \hookrightarrow \hat{D}$

$\downarrow$   
 $Poset \hookrightarrow D$

i.e. free directed completions of  
 $v$ -lattices, but maps

$$KL(\hat{D})(A, B) = Alg(S)(A, D(B))$$

are linear i.e. preserve  $\vee$ s.

Take identity-on-objects / full & f'ful  
factorization of  $KL(\hat{D}) \longrightarrow KL(D)$ .

$$AlgLatt(A, B) = Poset(A, D(B)).$$

# CARTESIAN CLOSURE

## Products

$$\begin{aligned}\text{Alg latt}(C, A \times B) &= \text{Poset}(C, D(A \times B)) \\ &= \text{Poset}(C, D(A) \times D(B)) \\ &= \text{Poset}(C, D(A)) \times \text{Poset}(C, D(B)) \\ &= \text{Alg latt}(C, A) \times \text{Alg latt}(C, B)\end{aligned}$$

So  $A \times B$  gives the product.

## Function space

$$\begin{aligned}\text{Alg latt}(A \times B, C) &= \text{Poset}(A \times B, DC) \\ &= \text{Poset}(A, B \Rightarrow DC)\end{aligned}$$

So need to show that

$$\begin{aligned}B \Rightarrow D(C) \\ \text{is of the form} \\ D(\cdot)\end{aligned}$$

NOT OBVIOUS!

# JUSTIFICATION

First suppose  $C = SZ$  is free on a poset  $Z$ . Then

$$\begin{aligned} B \Rightarrow DC &= B \Rightarrow PZ = (B \Rightarrow (Z^{\text{op}} \Rightarrow Z)) \\ &= (B^{\text{op}} \times Z)^{\text{op}} \Rightarrow Z = D(S(B^{\text{op}} \times Z)) \end{aligned}$$

(Also more or less obvious with bare hands.)

General case (sketch)

Any  $C$  lies in a  $U$ -split coequalizer diagram

$$S^2C \rightrightarrows SC \longrightarrow C$$

Using the  $\otimes$  (Kock c 1970) in  $S\text{-Alg}$  get a  $U$ -split fork

$$\begin{array}{ccc} S(B^{\text{op}} \times SC) & \rightrightarrows & S(B^{\text{op}} \times C) \\ \parallel & & \parallel \\ SB^{\text{op}} \otimes SC & \rightrightarrows & SB^{\text{op}} \otimes C \end{array}$$

Such colimits created (Beck 196?).

So obvious retract in  $\text{Posets}$  gives a  $v$ -semilattice. Then easy.

# AIMS REVISITED

- (a) Advancing other parts of computer science.

Variety of models      Extensions  
   Styles

Categorical models (Concurrency+)

- (b) Understanding the mathematical character of Domain Theory

Comparisons by mathematical underpinnings

- (c) Analysis of a general concept of approximation

ADT      SDT      Stone Duality

Applications to numerical analysis.

- (d) Other uses

- formal theories of fixed points ✓  
(eg Freyd)

# CONCLUSION

Kreisel :

- to correct two common and almost equally frustrating errors:

the first is to suppose that there is just one use of G.R.T. ....

the other is to suppose that there are so many ... that one does not even mention them.

Apply this mutatis mutandis to

Domain Theory