SOME REASONS FOR GENERALIZING DOMAIN THEORY

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DOMAINS IX

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CONTEXT

G. Kreisel. Some reasons for generalizing recursion theory

In Proceedings of the Summer School and Colloquium in Mathematical Logic, Manchester 1969

(Editors: Gandy, Yates)

Published 1971

AIMS

as discussed by Kreisel

(a) Advancing other parts of logic

Finiteness

Definability (Admissible sets)

(b) Understanding the mathematical character of ordinary recursion theory

Axiomatics (Wagner-Strong)

(c) Analysis of the general concept of computation

Church's Thesis Turing

(d) Other uses:

- formal theory of self application (Scott's D_∞)
- axiomatic analysis of kinds of evidence ?

(Categorical logic: Stone Duality?)

AIMS

re Domain Theory

(a) Advancing other parts of computer science

(b) Understanding the mathematical character of Scott's Domain Theory

(c) Analysis of a general concept of approximation

(d) Other uses

GENERALIZATIONS

of Domain Theory

Extensions of the category (SFP)

Other styles of domains (stable)

But for today

$\boxed{\text{Domains as categories}}$

Early work Taylor

Lamarche

Hyland - Pitts

(At the meeting I was reminded of Lehmann perhaps the earliest.)

Recent work Cattani - Winskel

Fiore - Gambino - Hyland - Winskel

LINEAR LOGIC

as potential distraction

Girard translation

$$\begin{aligned} \mathbb{C}(X \times Y, Z) &= \mathbb{L}(!(X \times Y), Z) \\ &= \mathbb{L}(!X \otimes !Y, Z) \\ &= \mathbb{L}(!X, !Y \multimap Z) \\ &= \mathbb{C}(X, !Y \multimap Z) \end{aligned}$$

While (by modified realizability) any cartesian closed category can be obtained from a model of classical linear logic thus,

this is NOT the reason why

$\boxed{\text{Algebraic Domains are Cartesian Closed}}$

RELATIONAL MODEL

for Linear Logic

Rel is compact closed and multisets M gives an exponential comonad

Rel together with the cartesian closed Kl(M) (Kleisli category) is very rich: it models

differential lambda calculus

fixed points

recursive domain equations

BUT it isn't domain theory

EXPLANATIONS

Rel is the Kleisli category $Kl(P)$ where

$$P(X) = (X \Rightarrow \Omega)$$

is the power set monad.

So compact closed and self dual

M multisets is the monad for commutative monoids on Sets and thus extends to Rel WHY?

Duality turns the monad to a comonad.

DISTRIBUTIVITY

M and P monads on $\mathcal{S}$

TFAE

①
$$\begin{tikzcd} M\text{-}Alg \arrow[loop right, "\hat{P}"] \arrow[d] \\ \mathcal{S} \arrow[loop right, "P"] \end{tikzcd}$$
P lifts to $M\text{-}Alg$

② $MP \xrightarrow{\lambda} PM$

distributive law

③
$$\begin{tikzcd} Kl(P) \arrow[loop right, "\tilde{M}"] \\ \mathcal{S} \arrow[loop right, "M"] \arrow[u] \end{tikzcd}$$
M extends to $Kl(P)$

and then PM a monad $Kl(PM) \simeq Kl(\tilde{M})$

(Analogous results for comonad/monad also important in CS.)

OPERADS
(linear monads)

For any cocomplete smc category $\mathcal{C}$, $CMon(\mathcal{C}) \longrightarrow \mathcal{C}$ is monadic, i.e. the free commutative monoid monad M exists.

The power set P is commutative (it is $\vee$-lattices) so monoidal and so lifts to $CMon(\mathcal{C}) = M\text{-}Alg$

$\therefore$ M extends to $K\ell(P)$

Same with M replaced by any operadic = linear monad.

Folklore? Eg. Eilenberg-Wright

PROFUNCTORS

(Winskel's approach to concurrency)

$P(\mathbb{A}) = [\mathbb{A}^{op}, Set]$ category of processes of shape.

$$Prof(\mathbb{A}, \mathbb{B}) = [\mathbb{A} \times \mathbb{B}^{op}, Set]$$
$$= [\mathbb{A}, P(\mathbb{B})]$$

category of set-valued relations from $\mathbb{A}$ to $\mathbb{B}$.

Such a $\mathbb{A} \xrightarrow{F} \mathbb{B}$ induces (action on points)

$$P(\mathbb{A}) = [\mathbb{A}^{op}, Set] \longrightarrow [\mathbb{B}^{op}, Set] = P(\mathbb{B})$$

linear map on processes

Non-linear maps from (pseudo) comonads which arise by extending monads on Cat to $Kl(P) = Prof$

LIFTS/EXTENSIONS

Fix $P\mathbb{A} = [\mathbb{A}^{op}, \mathrm{Set}]$ (not quite a monad but never mind!) and M a 2-monad on $\mathcal{C}at$

$$\begin{array}{ccccc} & M\text{-alg} \circlearrowleft \hat{P} & & Kl(P) = \mathrm{Prof} \circlearrowleft \tilde{M} & \\ P \text{ lifts} & \downarrow & \text{iff} & \uparrow & M \text{ extends} \\ & \mathcal{C}at \circlearrowleft P & & \mathcal{C}at \circlearrowleft & \end{array}$$

This means good behaviour in

$$\begin{tikzcd} \mathbb{A} \arrow[r, "y"] \arrow[dr, "f"'] & P(\mathbb{A}) \arrow[d, "\mathrm{Lan}_y f"] \\ & \mathcal{E} \end{tikzcd}$$

for M-algebras.

EXAMPLES

of suitable monads

M terminal object

products

finite limits

(symmetric) monoidal

Resulting comonad adds finite colimits – superficially analogous to domain theory

Models differential λ-calculus with fixed points: Joyal's species are in $Kl(\tilde{M})(1,1)$

IND-COMPLETION

For $\mathbb{A}$ small we have the Yoneda embedding $y : \mathbb{A} \longrightarrow P(\mathbb{A}) = [\mathbb{A}^{op}, Set]$

Every $X \in P(\mathbb{A})$ is the colimit over its category of elements $\mathbb{E}(X)$ of representables.

Let $D(\mathbb{A})$ consist of the X with $\mathbb{E}(X)$ filtered.

$P(\mathbb{A})$ closure of $\mathbb{A}$ under colimits

$D(\mathbb{A})$ closure of $\mathbb{A}$ under filtered colimits

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, hook] \arrow[dr, hook] & & P(\mathbb{A}) \\
& D(\mathbb{A}) \arrow[ur, hook] &
\end{tikzcd}$$

FINITE COLIMITS

If $\mathbb{C}$ small with finite colimits
then $D(\mathbb{C})$ has finite colimits
(so is cocomplete) and

$$\mathbb{C} \longrightarrow D(\mathbb{C})$$

preserves finite colimits.

Also in

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[dr, "f"'] & D(\mathbb{C}) \arrow[d, "\hat{f}"] \\ & \mathcal{X} \end{tikzcd}$$

if f preserves finite colimits then
so does $\hat{f}$ (so it is cocontinuous)

LIFTS / EXTENSIONS

for the Ind Completion

We know

D lifts to $S\text{-}Alg$

$$\begin{tikzcd} S\text{-}Alg \arrow[loop right, "\hat{D}"] \arrow[d] \\ Cat \arrow[loop right, "D"] \end{tikzcd}$$

Hence

S extends to $Kl(D)$

$$\begin{tikzcd} Kl(D) \arrow[loop right, "\tilde{S}"] \\ Cat \arrow[loop right, "S"] \arrow[u] \end{tikzcd}$$

Moreover DS a monad $\simeq P$

and $Kl(\tilde{S}) \simeq Kl(DS) \simeq Kl(P) = Prof$

Generalized domain theory can be extracted from this setting.

POSET CASE

D monad for directed completion

S monad for $\vee$-slattices

$DS = P$ the "power set" monad

$$A \longmapsto (A^{op} \Rightarrow 2)$$

on posets

Then

D lifts to $\vee$-slattices $= S\text{-}Alg$

S extends to $Kl(D)$ category of (directed relations or) free dcpos and Scott continuous maps

DCPOs

This is the category D-Alg

D is a (necessarily) commutative monad with

$$D(A \times B) \xrightarrow{\simeq} D(A) \times D(B)$$

the inverse of the monoidal structure.

Thus D-Alg is cartesian closed.

ALGEBRAIC LATTICES

Objects are easy $Kl(\hat{D})$ from the lift

$$\begin{tikzcd} \mathrm{Alg}(S) \arrow[loop right, "\hat{D}"] \arrow[d] \\ \mathrm{Poset} \arrow[loop right, "D"] \end{tikzcd}$$

i.e. free directed completions of $\vee$-slattices, but maps

$$Kl(\hat{D})(A,B) = \mathrm{Alg}(S)(A, D(B))$$

are linear i.e. preserve $\vee$s.

Take identity-on-objects / full & f'ful factorization of $Kl(\hat{D}) \longrightarrow Kl(D)$.

$$\boxed{\mathrm{AlgLatt}(A,B) = \mathrm{Poset}(A, D(B))}$$

CARTESIAN CLOSURE

Products

$$\mathrm{AlgLatt}(C, A\times B) = \mathrm{Poset}(C, D(A\times B))$$
$$= \mathrm{Poset}(C, D(A)\times D(B))$$
$$= \mathrm{Poset}(C, D(A))\times \mathrm{Poset}(C, D(B))$$
$$= \mathrm{AlgLatt}(C,A)\times \mathrm{AlgLatt}(C,B)$$

So $A\times B$ gives the product.

Function space

$$\mathrm{AlgLatt}(A\times B, C) = \mathrm{Poset}(A\times B, DC)$$
$$= \mathrm{Poset}(A, B \Rightarrow DC)$$

So need to show that

$$B \Rightarrow D(C)$$

is of the form

$$D(\cdot)$$

NOT OBVIOUS!

JUSTIFICATION

First suppose $C = SZ$ is free on a poset Z. Then

$$B \Rightarrow DC = B \Rightarrow PZ = (B \Rightarrow (Z^{op} \Rightarrow 2))$$
$$= (B^{op} \times Z)^{op} \Rightarrow 2 = D(S(B^{op} \times Z))$$

(Also more or less obvious with bare hands.)

General case (sketch)

Any C lies in a U-split coequalizer diagram

$$S^2C \rightrightarrows SC \longrightarrow C$$

Using the $\otimes$ (Kock c 1970) in S-Alg get a U-split fork

$$\begin{array}{ccc} S(B^{op} \times SC) & \rightrightarrows & S(B^{op} \times C) \\ \| & & \| \\ SB^{op} \otimes S^2C & \rightrightarrows & SB^{op} \otimes SC \end{array}$$

Such colimits created (Beck 196?).

So obvious retract in Posets gives a $\vee$-semilattice. Then easy.

AIMS REVISITED

(a) Advancing other parts of computer science.

Variety of models Extensions
Styles

$\boxed{\text{Categorical models}}$ (Concurrency +)

(b) Understanding the mathematical character of Domain Theory

$\boxed{\text{Comparisons by mathematical underpinnings}}$

(c) Analysis of a general concept of approximation

ADT SDT Stone Duality

Applications to numerical analysis.

(d) Other uses

- formal theories of fixed points

(eg Freyd)

CONCLUSION

Kreisel:

- to correct two common and almost equally frustrating errors:
  the first is to suppose that there is just one use of G.R.T. .....
  the other is to suppose that there are so many ... that one does not even mention them.

Apply this mutatis mutandis to

Domain Theory