STONE DUALITY

Theories dual to **Categories of models**

$$I : T_1 \longrightarrow T_2 \quad \text{an interpretation}$$

$\Downarrow$ $\Uparrow$ — "strong completeness"

$$\mathrm{Mod}(I) : \mathrm{Mod}(T_2) \longrightarrow \mathrm{Mod}(T_1)$$

Why?

$$\mathrm{Mod}(T) \text{ "="} \{ J : T \longrightarrow S \}$$

so that

$$\mathrm{Mod}(I) : \mathrm{Mod}(T_2) \longrightarrow \mathrm{Mod}(T_1)$$

$$: (J : T_2 \longrightarrow S) \longmapsto (J \circ I : T_1 \longrightarrow S).$$

STRICT HORN PROPOSITIONAL LOGIC

PROP$^{\underline{L}}$ SYMBOLS $p, q, \ldots$

"FRAGMENT" closed under $T, \wedge$

"AXIOMS" $\phi \vdash \psi$ (ϕ, ψ from fragment)

(n.b. can assume of form $\phi \vdash p$)

Lindenbaum algebra:

a $\wedge$- semilattice

($\sim$ $\sqcup$-semi-lattice of finite elements)

(n.b. T = true corresponds to $\bot$ = bottom)

Models ("points") of theory:

filters

($\sim$ ideals = elements of domain)

Space of models:

an algebraic lattice

(with Scott topology)

Write $\overline{\Phi}$ for element of domain corresponding to $\phi \in \wedge$-semi-lattice and $O_\phi = \{ x \mid \overline{\Phi} \sqsubseteq x \}$.

Infinitary propositions:

$$\bigvee_i \phi_i$$ (can assume $\downarrow$-closed)

~ open sets

$$\bigcup_i O_{\phi_i}$$ (can assume $\uparrow$-closed)

For x a point $\quad x \models \bigvee_i \phi_i$

$$\sim \quad x \in \bigcup_i O_{\phi_i}$$

(regarding x as filter / ideal this is $x \cap \{\phi_i\}$ non-empty)

$$\phi \vdash \psi \quad (\phi \leq \psi) \quad \sim \quad \overline{\Phi} \sqsupseteq \overline{\Psi}$$

$$\text{Homomorphism } x \longrightarrow y \quad \sim \quad x \sqsubseteq y$$

SPECIAL FEATURE: finite elements have dual nature –

(1) they are special elements of domain = points = models of theory

(2) they are special open sets (in fact a basis of such) = propositions

(In fact as models have the free model generated by the data of the proposition. Whence completeness theorem trivial – essentially algebraic.)

THE POWER SET OF A SET

A a set

For each $a \in A$ take a proposition $a \in X$.

Theory: (no axioms)

Space of models:

power set of A

Open subsets of Cantor space

For each finite sequence u of 0s and 1s take a proposition

$$u \in O$$

Theory:

$$u*0 \in O \wedge u*1 \in O \vdash u \in O$$

Space of models:

open subsets of Cantor space

HORN PROPOSITIONAL LOGIC

PROPL SYMBOLS $p, q, \ldots$

"FRAGMENT" closed under $\top, \wedge, \perp$

"AXIOMS" $\phi \vdash \psi$ (ϕ, ψ from fragment)

(n.b. can assume of form $\phi \vdash p$, $\phi \vdash \perp$)

"Lindenbaum algebra" (forgetting $\perp$)

a $\wedge$-"hemi"lattice (i.e. consistent $\wedge$s exist)

($\sim$ "consis-$\vee$"-semi-lattice of finite elements of a "consistently complete" domain)

Models ("points") of theory:

filters

($\sim$ ideals = elements of domain)

Space of models:

directedly complete + consistently complete (with $\perp$)

algebraic poset

Finite and infinite words

A a set

For each finite sequence u in A take a proposition $u \subset x$.

Theory:

$$\vdash \langle\rangle \subset x$$

$$u \subset x \vdash v \subset x \qquad (v \subset u)$$

$$u \subset x \wedge v \subset x \vdash \bot \qquad (u, v \text{ are "incompatible"})$$

Space of models

domain of words (lists)

For total elements add

$$u \subset x \vdash \bigvee_{a \in A} u * a \subset x$$

Approximations in the completion of a local ring.

R a local ring, $\mathfrak{m}$ the unique maximal ideal.

For each $a \in R$ $k \in \mathbb{N}$ take a proposition $x =_k a$ (i.e. $x = a$ $(\mathfrak{m}^k)$)

Theory:

$$x =_k a \vdash x =_k b \qquad (a-b \in \mathfrak{m}^k)$$

$$x =_{k+1} a \vdash x =_k a$$

$$x =_k a \wedge x =_k b \vdash \bot \qquad (a-b \notin \mathfrak{m}^k)$$

Space of models:

domain of the completion

For total elements add

$$\vdash x =_0 1$$

$$x =_k a \vdash \bigvee_{a'} x =_{k+1} a'$$

"Approximable Relations"

A, B finite elements in domains $\hat{A}$, $\hat{B}$

For $a \in A$ $b \in B$ take a proposition $R(a,b)$ (" $b \sqsubseteq f(a)$ ")

Theory: $R(a,b) \vdash R(a',b')$

$(a \sqsubseteq a', b \sqsupseteq b')$

$\vdash R(a,\bot)$

$R(a,b_1) \wedge R(a,b_2) \vdash R(a, b_1 \sqcup b_2)$

(if $\nearrow$ exists)

$R(a,b_1) \wedge R(a,b_2) \vdash \bot$ (o.w.)

Space of models

function space $[\hat{A} \longrightarrow \hat{B}]$

Remark: "profunctor" : $A^{op} \times B \longrightarrow \binom{\top}{\bot}$

"bimodule" (w.r.t. $\leq$)

which is flat in B.

CLASSICAL PROPOSITIONAL LOGIC

Lindenbaum algebra :

a boolean algebra

Space of models :

the Stone space

POSITIVE PROPOSITIONAL LOGIC

FRAGMENT : closed under $\top, \wedge, \bot, \vee$

Lindenbaum algebra :

a distributive lattice

Space of models :

the coherent space

THE SPECTRUM OF A RING

R a commutative ring (with a 1).

For each $a \in R$ take a proposition

$$\mathrm{inv}(a)$$

(for a is invertible).

Theory:

$$\vdash \mathrm{inv}(1)$$

$$\mathrm{inv}(a) \wedge \mathrm{inv}(b) \dashv\vdash \mathrm{inv}(ab)$$

$$\mathrm{inv}(0) \vdash \bot$$

$$\mathrm{inv}(a+b) \vdash \mathrm{inv}(a) \vee \mathrm{inv}(b)$$

Space of models:

$$\mathrm{Spec}(R).$$