GAME SEMANTICS

1. Games for computation:
partial strategies,
intuitively uniform features

2. Games and PCF:
fixed points,
intensional features

3. Games for logic:
total strategies,
non-uniform features

4. Dialogue games & arguments:
weak completeness

GENERAL FEATURES

- Computation or proof by data flow
- Uniform features (copying)
- Intensional features

$\downarrow$ Weak indicator

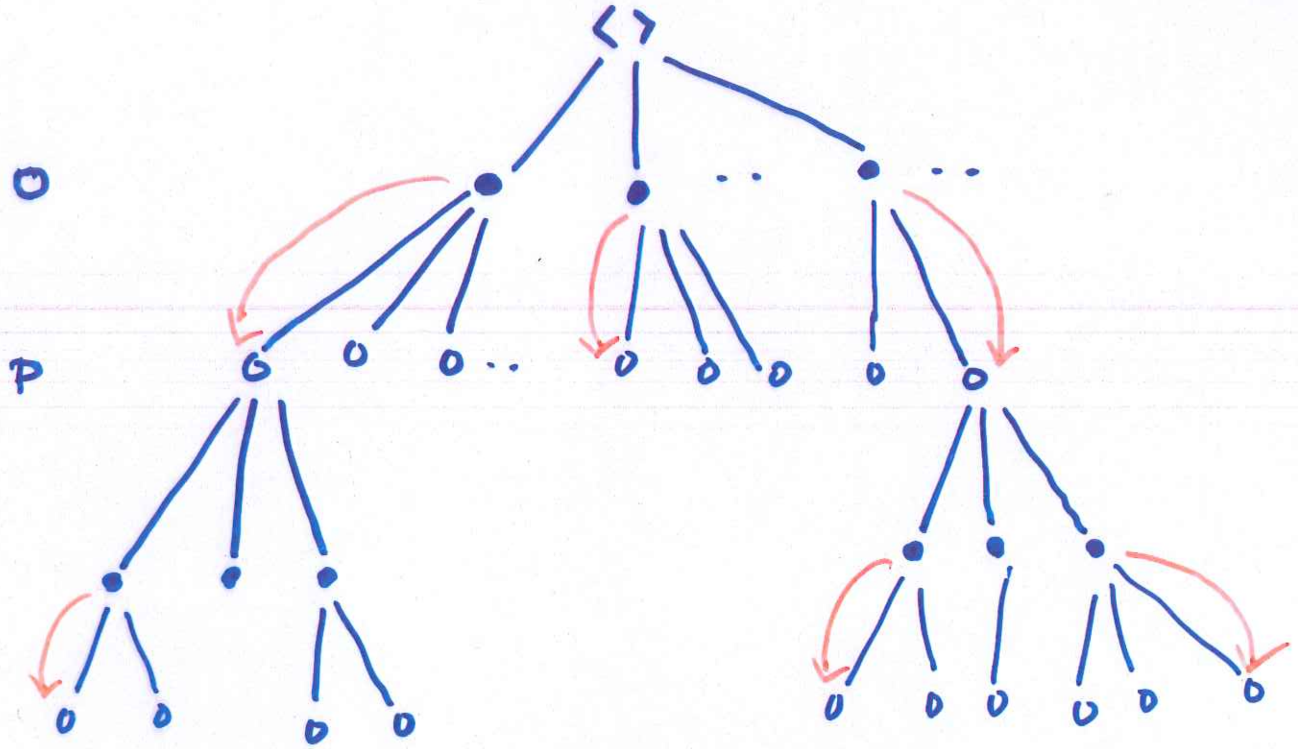
Failure of extensionality

<u>Extensionality</u>

Suppose $f, g : A \to B$ and for all $a : A$, $f(a) = g(a)$.

Then $f = g$.

Games & Strategies

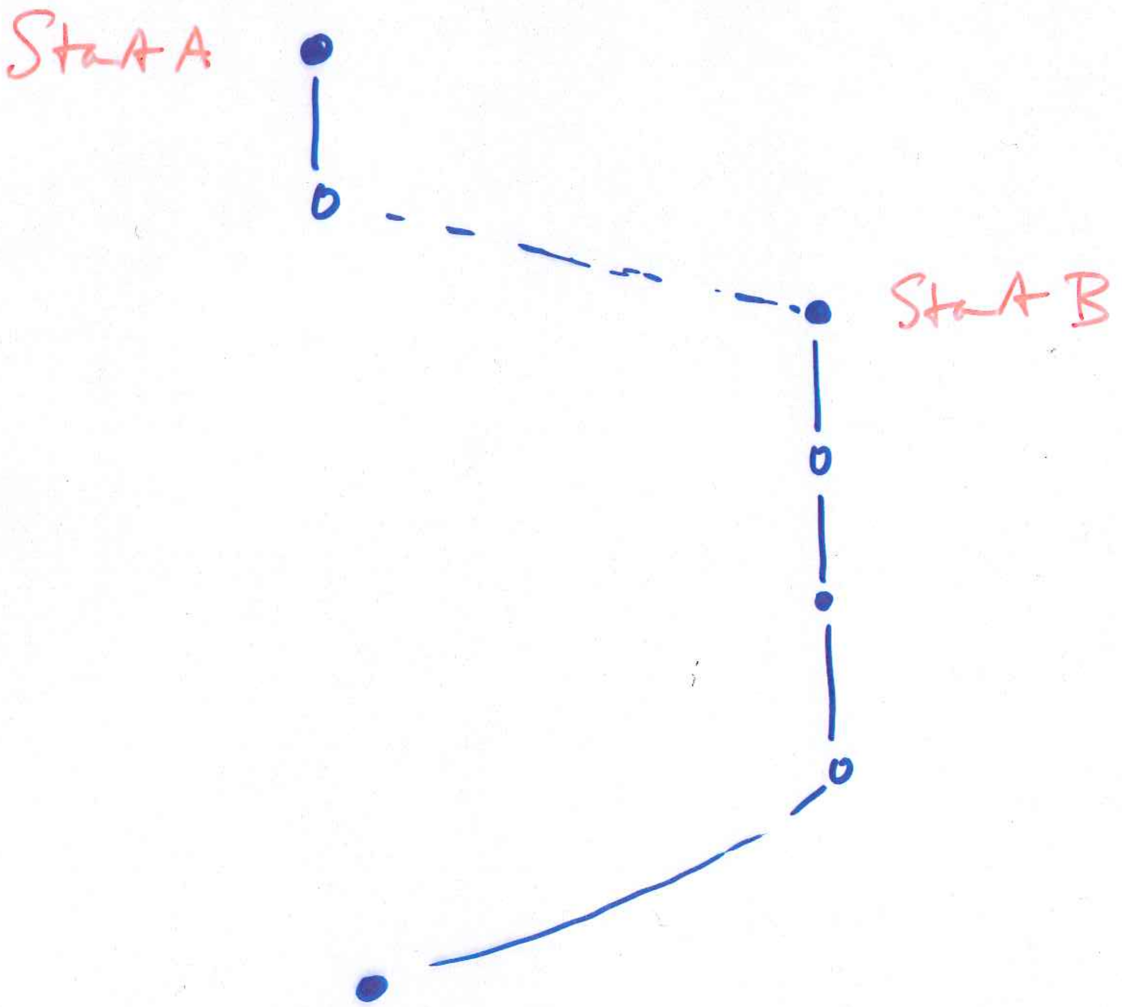


- Players strategies
- Opponent plays first
- Moves are labelled
- Strategies are determistic (but)
- Games may be infinite
- To start with strategies may be partial

GAMES IN //.

TENSOR

$$A \otimes B$$

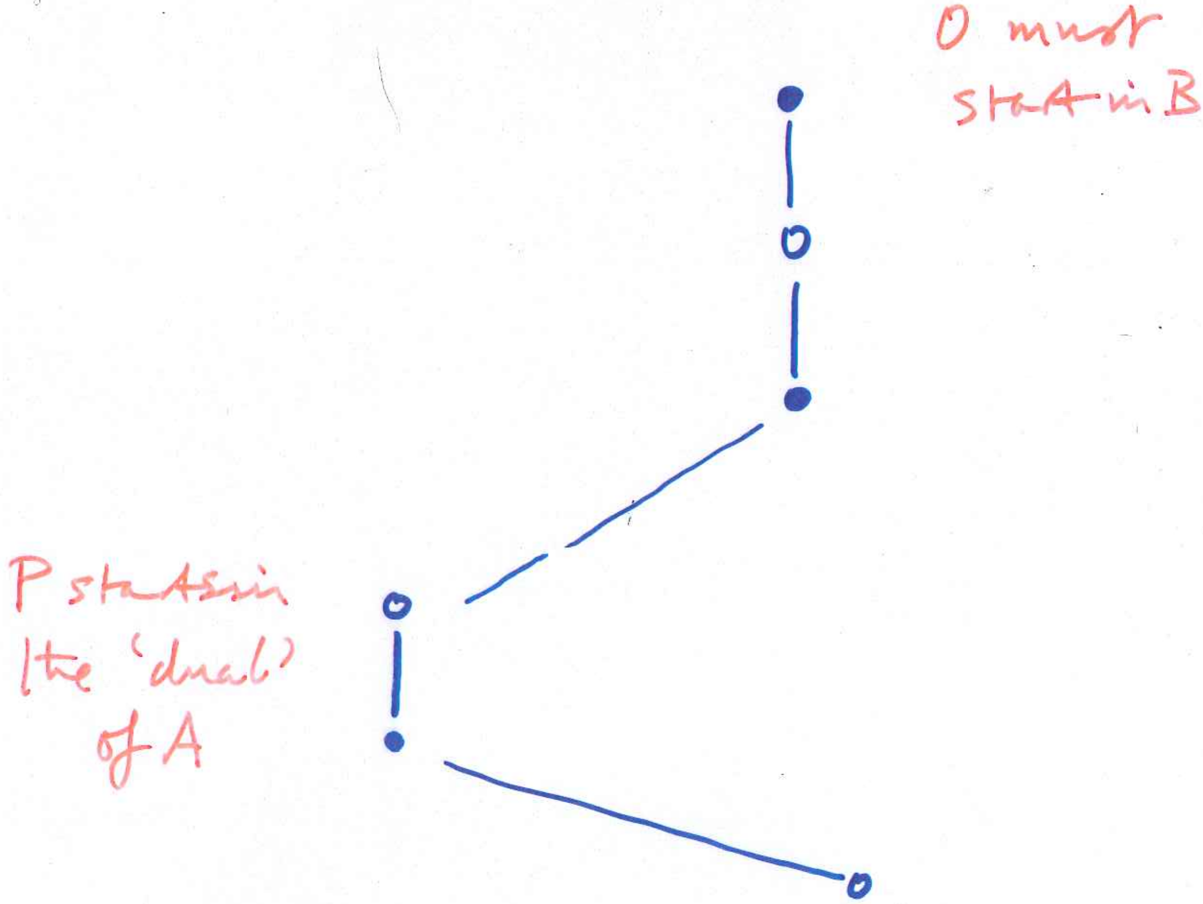


GAMES IN //

LINEAR FUNCTION SPACE

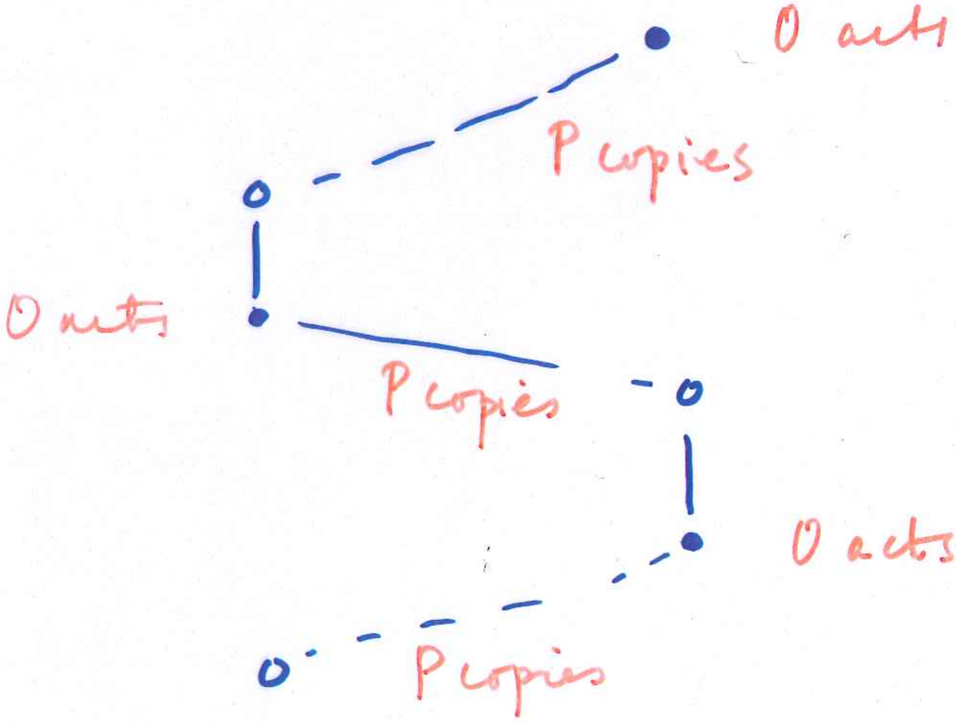
$$A \multimap B$$

$$A^{\perp} \parr B$$



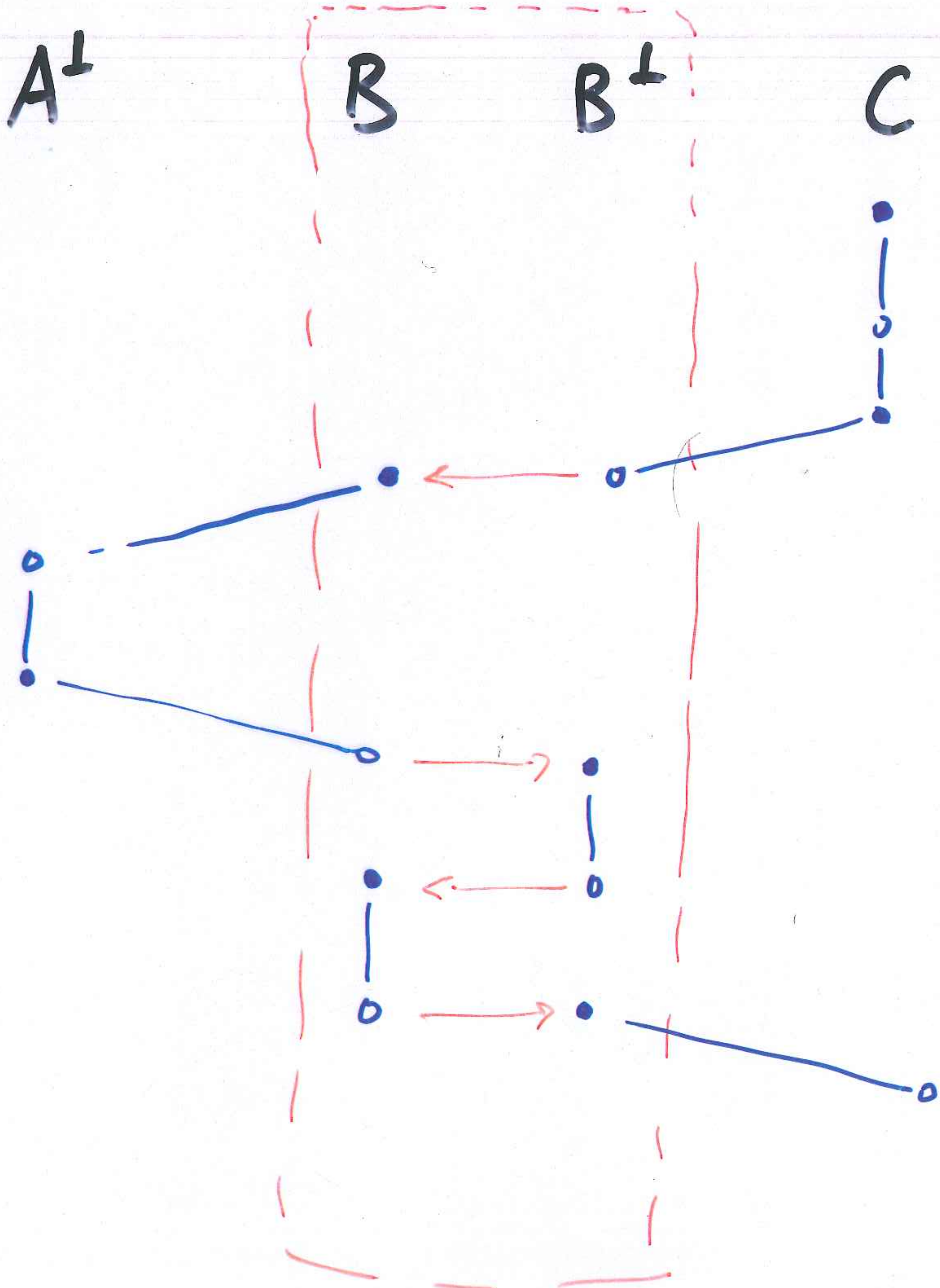
IDENTITY STRATEGY IN $A \multimap A$

$A^{\perp} \qquad A$



COMPOSITION

$$A \overset{\sigma}{\multimap} B \qquad B \overset{\tau}{\multimap} C$$



EXAMPLE

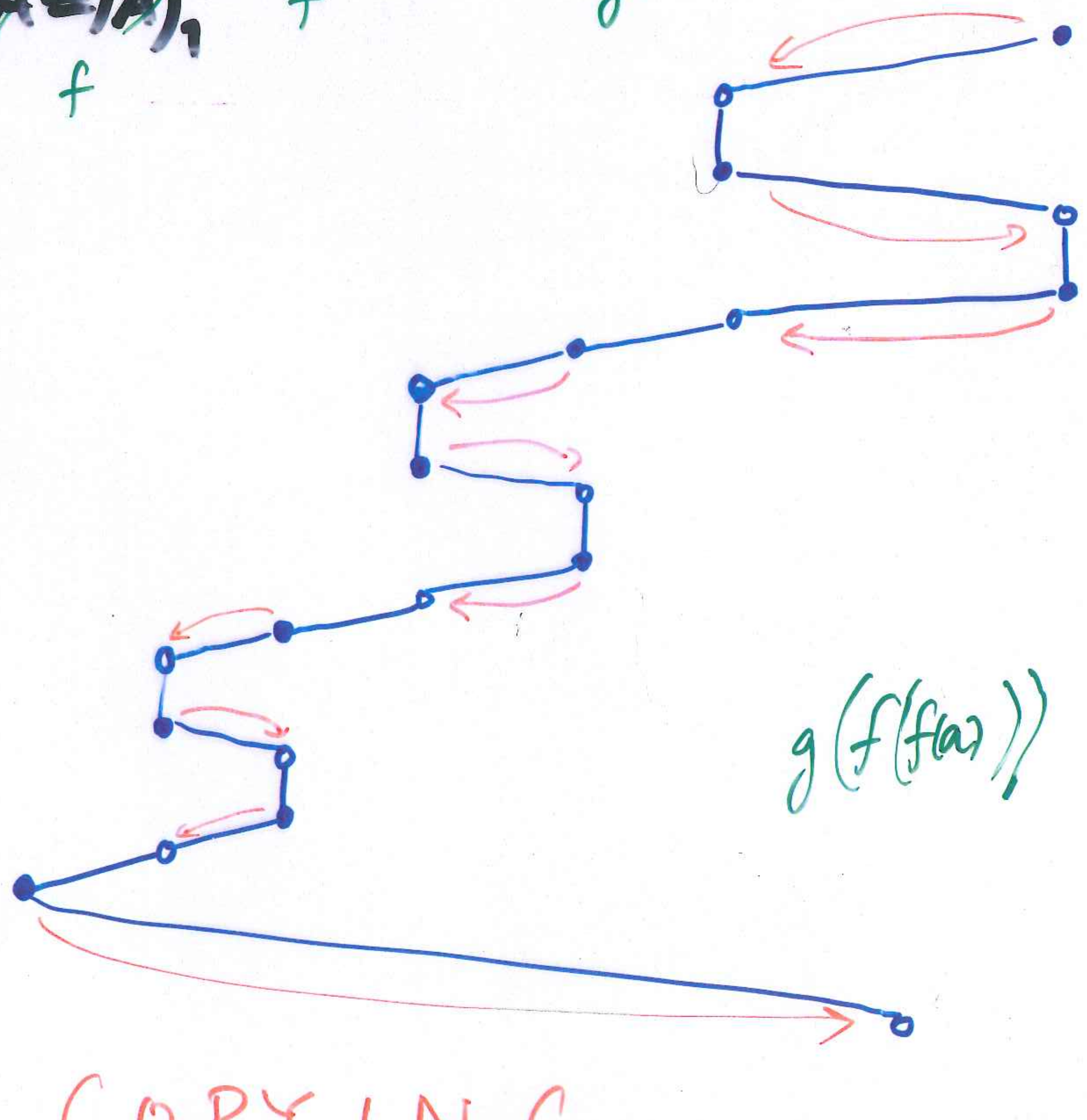
$$\lambda f : \cancel{A}^{B} \Rightarrow \cancel{A}^{B} .\ \lambda g : A \Rightarrow B .\ \lambda a : A .\ \cancel{f(f(g(a)))}^{\,g(f(f(a)))}$$

$$: (\cancel{A}^{B} \Rightarrow \cancel{A}^{B}) \Rightarrow ((\cancel{A}^{B} \Rightarrow B) \Rightarrow (A \Rightarrow B))$$

$\underset{f}{(\cancel{A}^{B} \Rightarrow \cancel{A}^{B})_0}$ $\quad \underset{g}{A \Rightarrow B}$ $\quad \underset{a}{A \Rightarrow B}$

$\underset{f}{(\cancel{A}^{B} \Rightarrow \cancel{A}^{B})_1}$

. .



$g(f(f(a)))$

COPYING

SYMMETRIC MONOIDAL CLOSED CATEGORY OF GAMES

Objects: Games $A, B, C, \ldots$

Maps: Strategies σ in $A \multimap B$ are maps $A \to B$

SMC Structure

$$A \otimes B \multimap C \quad \cong \quad A \multimap (B \multimap C)$$

$$A^{\perp} \parallel B^{\perp} \parallel C \qquad\qquad A^{\perp} \parallel B^{\perp} \parallel C$$

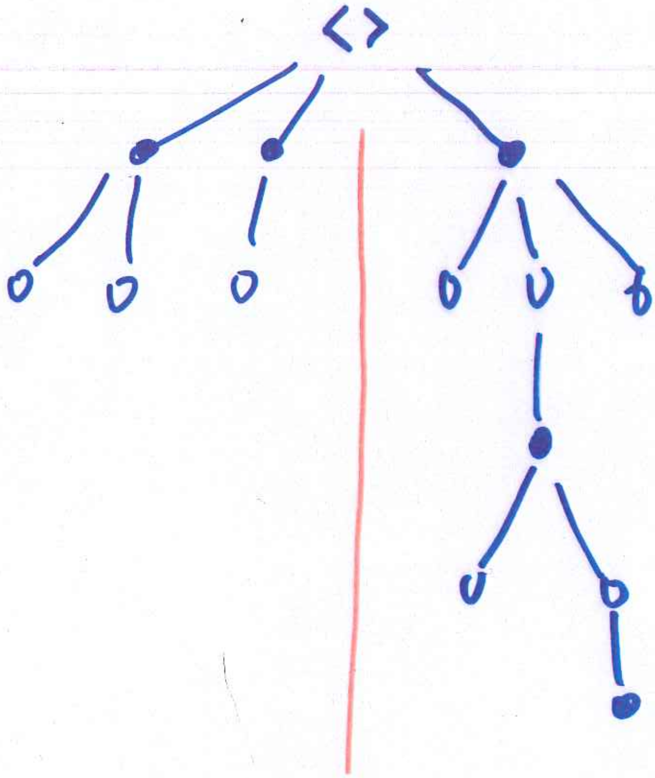
WARNING

First-order easy

Higher-order not so

PRODUCTS

$$A \times B$$



$$C \multimap A \times B$$

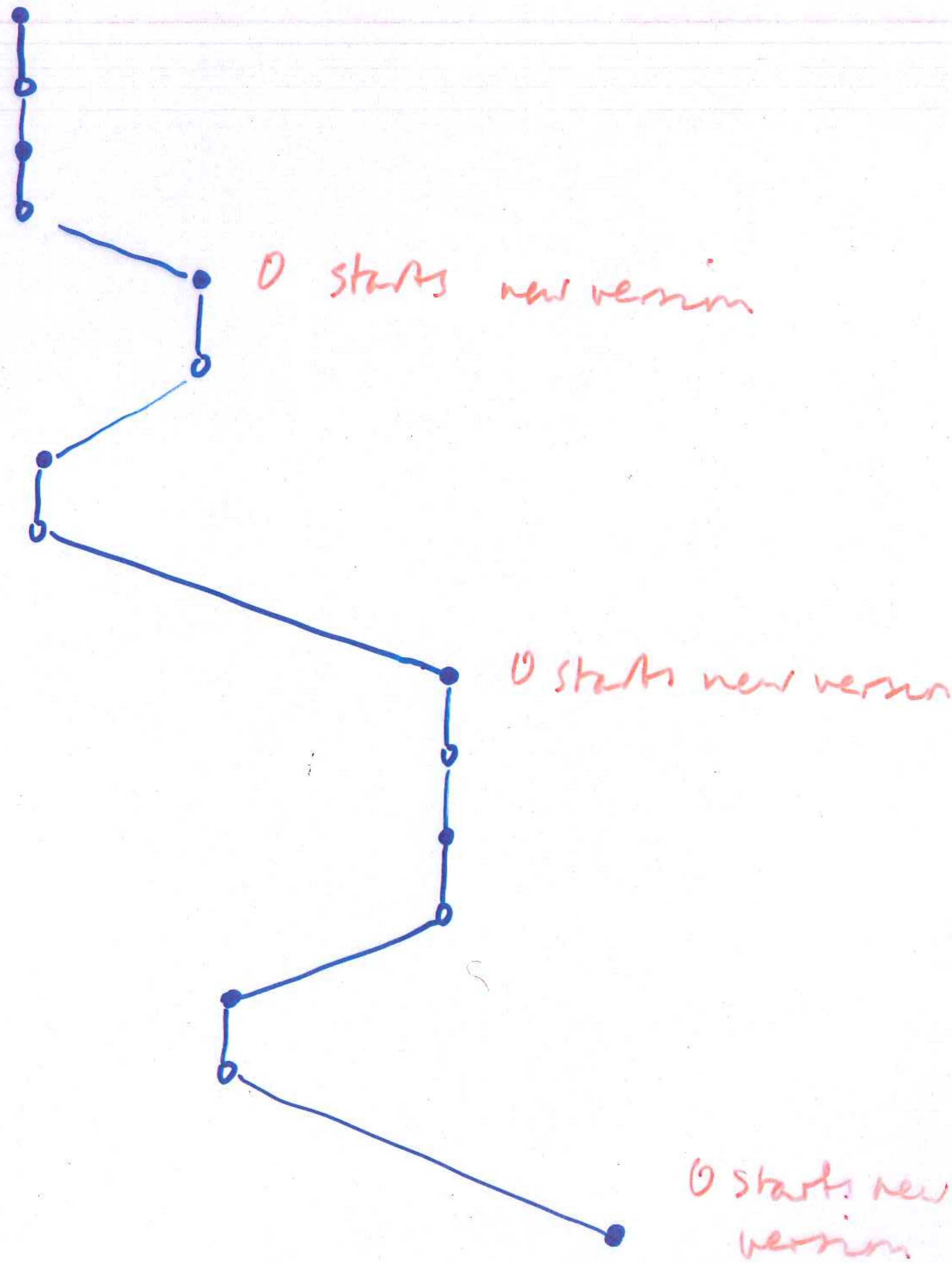
A strategy is a pair of

- a strategy in $C \multimap A$
- a strategy in $C \multimap B$

CONTROL

EXPONENTIAL COMONAD

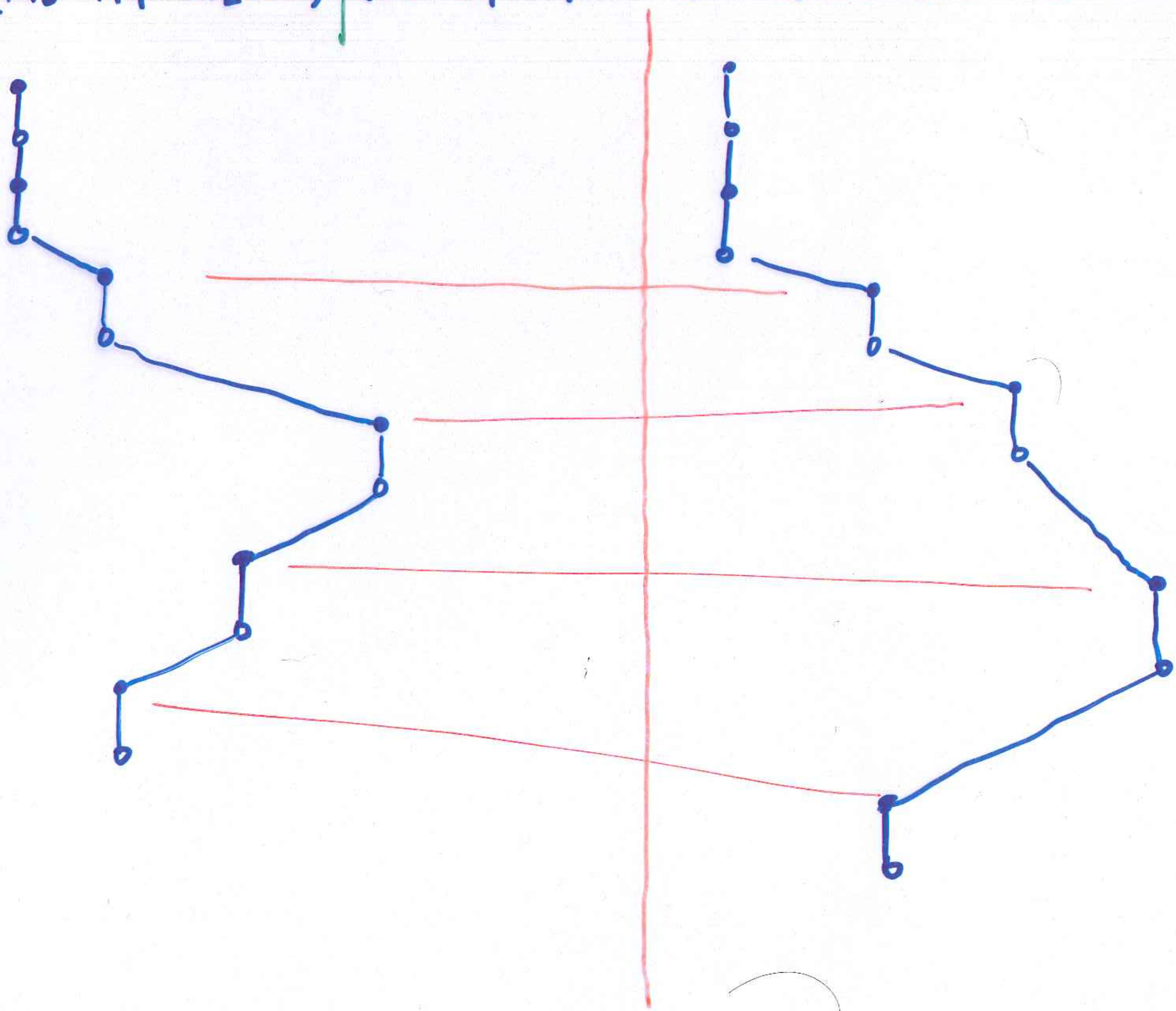
$$!A = A_0 \otimes A_1 \otimes A_2 \otimes \cdots\cdots$$



SEELY ISOMORPHISM

$$!A \otimes !B \cong !(A \times B)$$

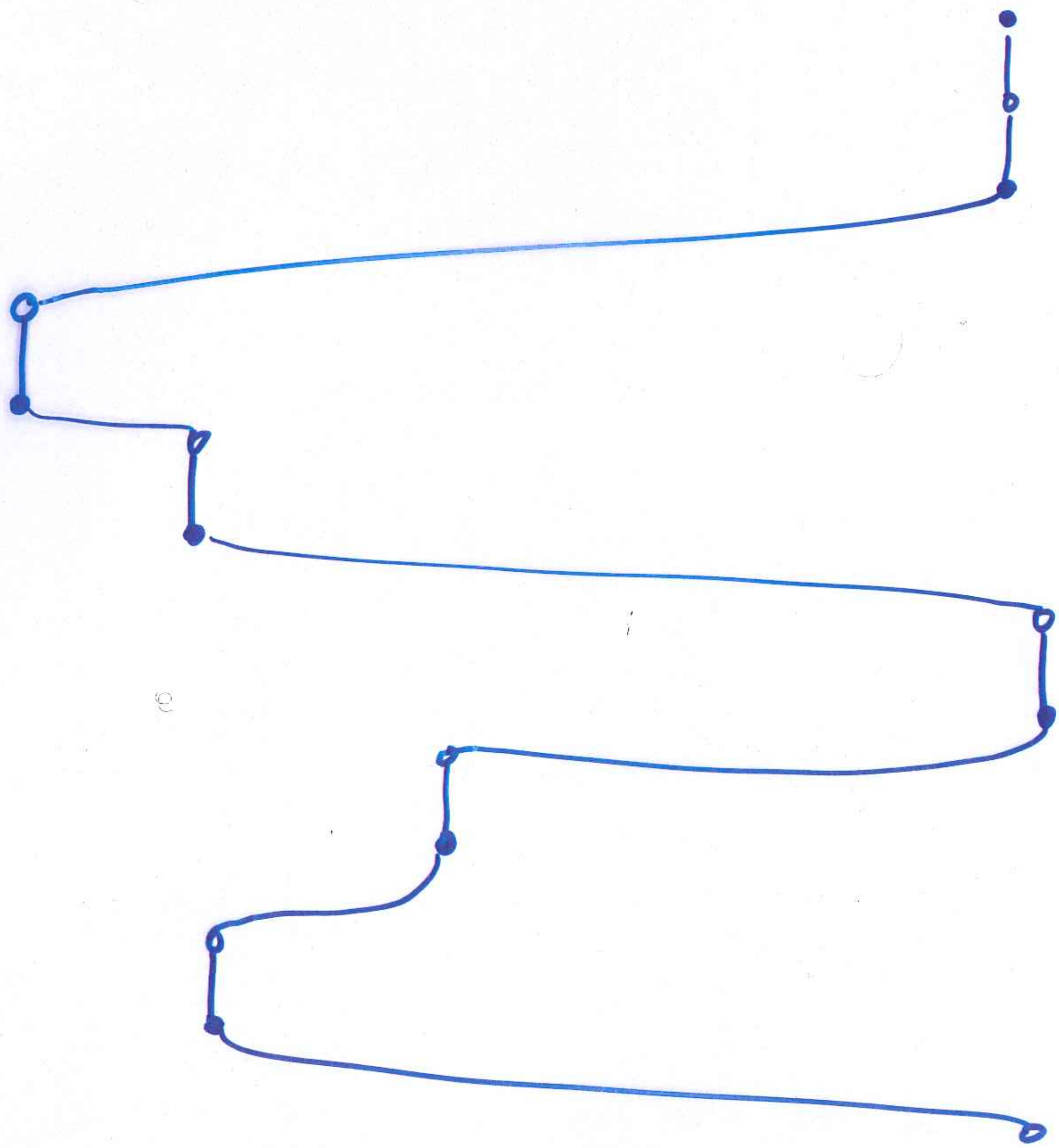
$|(A_0 \ A_1 \ A_2 \dots) \,|\, (B_0 \ B_1 \dots)$ $\qquad$ $(A\times B)_0 \ (A\times B)_1 \ (A\times B)_2 \ (A\times B$



GENERAL FUNCTION SPACE

$$(A \Rightarrow B) = (!A \multimap B)$$

$A_0^\perp \quad A_1^\perp \quad A_2^\perp \; \ldots \qquad B$

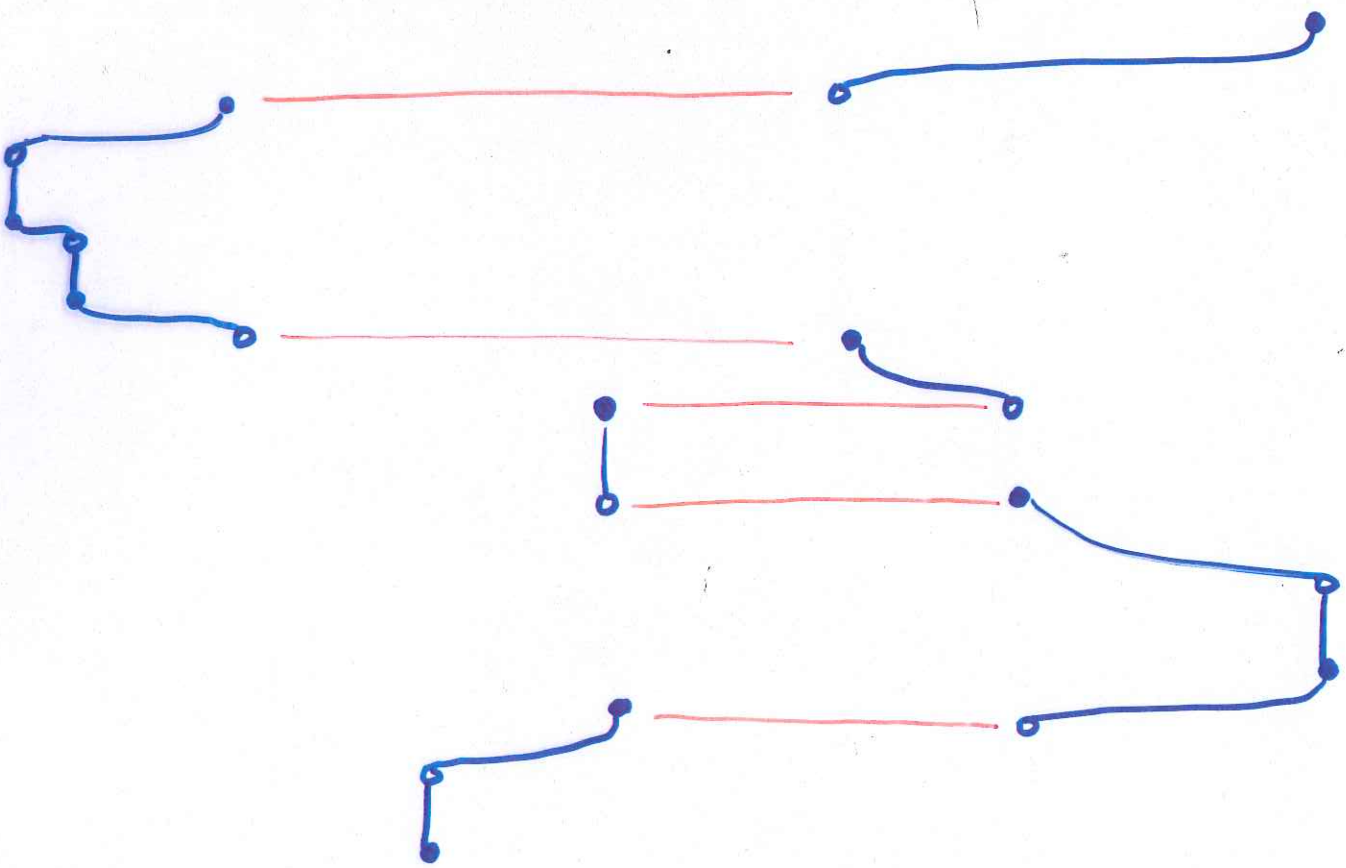


KLEISLI COMPOSITION

$$A \xRightarrow{\sigma} B \qquad B \xRightarrow{\tau} C$$

$(A \Rightarrow B)_0$ $\quad$ $B_0^{\perp}$ $\quad$ B_1^{+} .... $\quad$ C

$(A \Rightarrow B)_1$ ...



CARTESIAN CLOSED CATEGORY OF GAMES

Objects Games $A, B, C, \ldots$

Maps Strategies in $(A \Rightarrow B) = (!A \multimap B)$ are maps $A \longrightarrow B$.

CC Structure

$$
\begin{aligned}
(A \times B \Rightarrow C) &= \,!(A \times B) \multimap C \\
&\cong \,!A \otimes !B \multimap C \\
&\cong \,!A \multimap (!B \multimap C) \\
&= A \Rightarrow (B \Rightarrow C)
\end{aligned}
$$

Is There A (Natural) Isomorphism

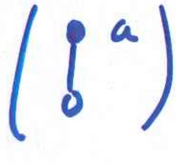
$$!A \otimes !B \cong !(A \otimes B)$$

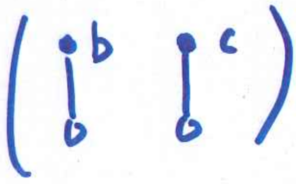
$$(A_0 \otimes A_1 \otimes A_2 \cdots) \otimes (B_0 \otimes B_1 \otimes B_2 \cdots)$$

$$\stackrel{?}{\cong} \quad (A \otimes B)_0 \otimes (A \otimes B)_1 \otimes (A \otimes B)_2 \cdots$$

Answer: NO

Not even isomorphic in case

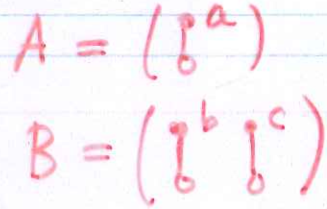
$A =$ 

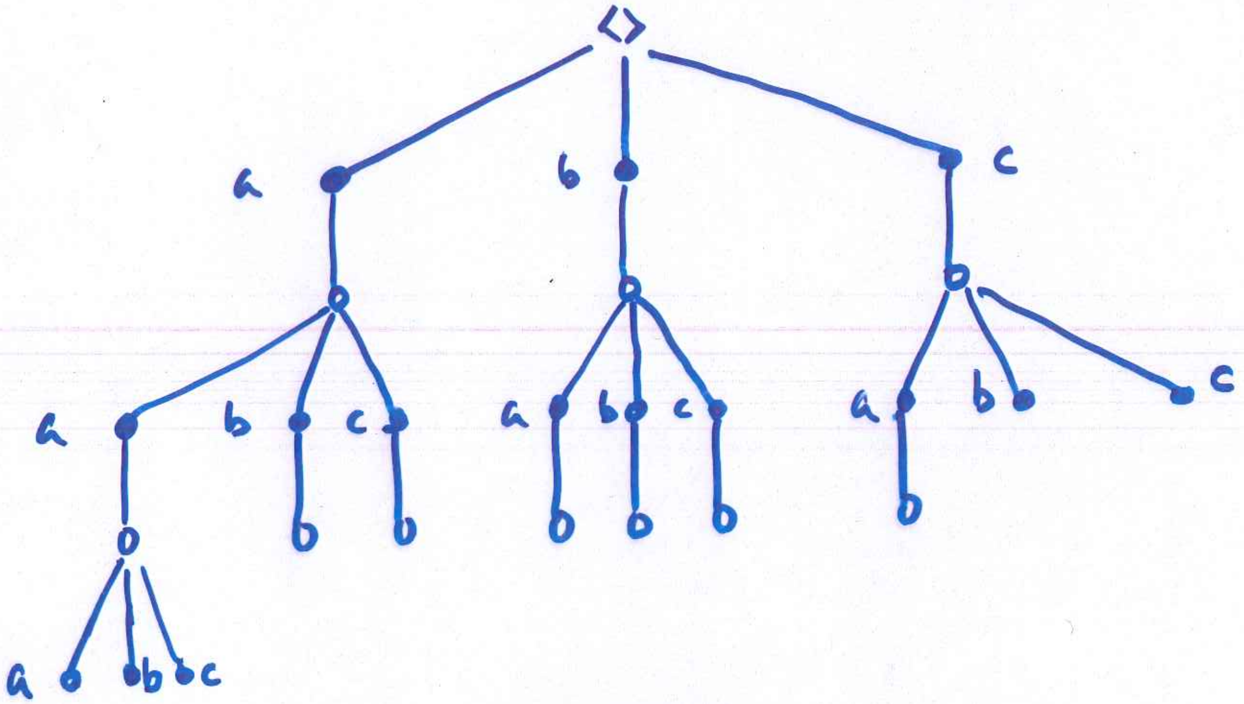
$B =$ 

P has no options

$(B \cong A \times A)$

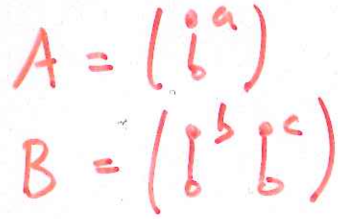
$!A \otimes !B$

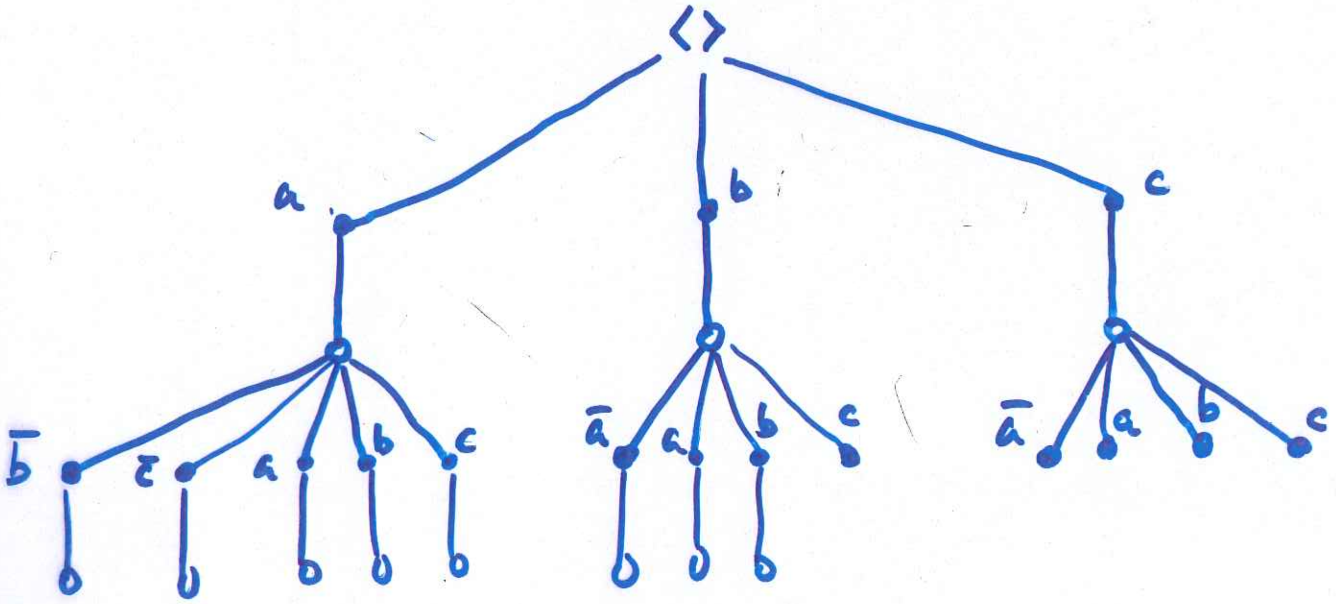




$$=$$

$!(A \otimes B)$





COMPUTATIONAL EXPLANATION

NICK BENTON :

$!(A \otimes B)$ 'obviously' not isomorphic to $!A \otimes !B$ because

- $!C$ is <u>not</u> an "infinite tensor product" but rather a "new copy of C for opponent on demand!"

Hence

- In $!(A \otimes B)$ Opponent always has the same number of copies of A and B available; but the same not true of $!A \otimes !B$

EXAMPLE

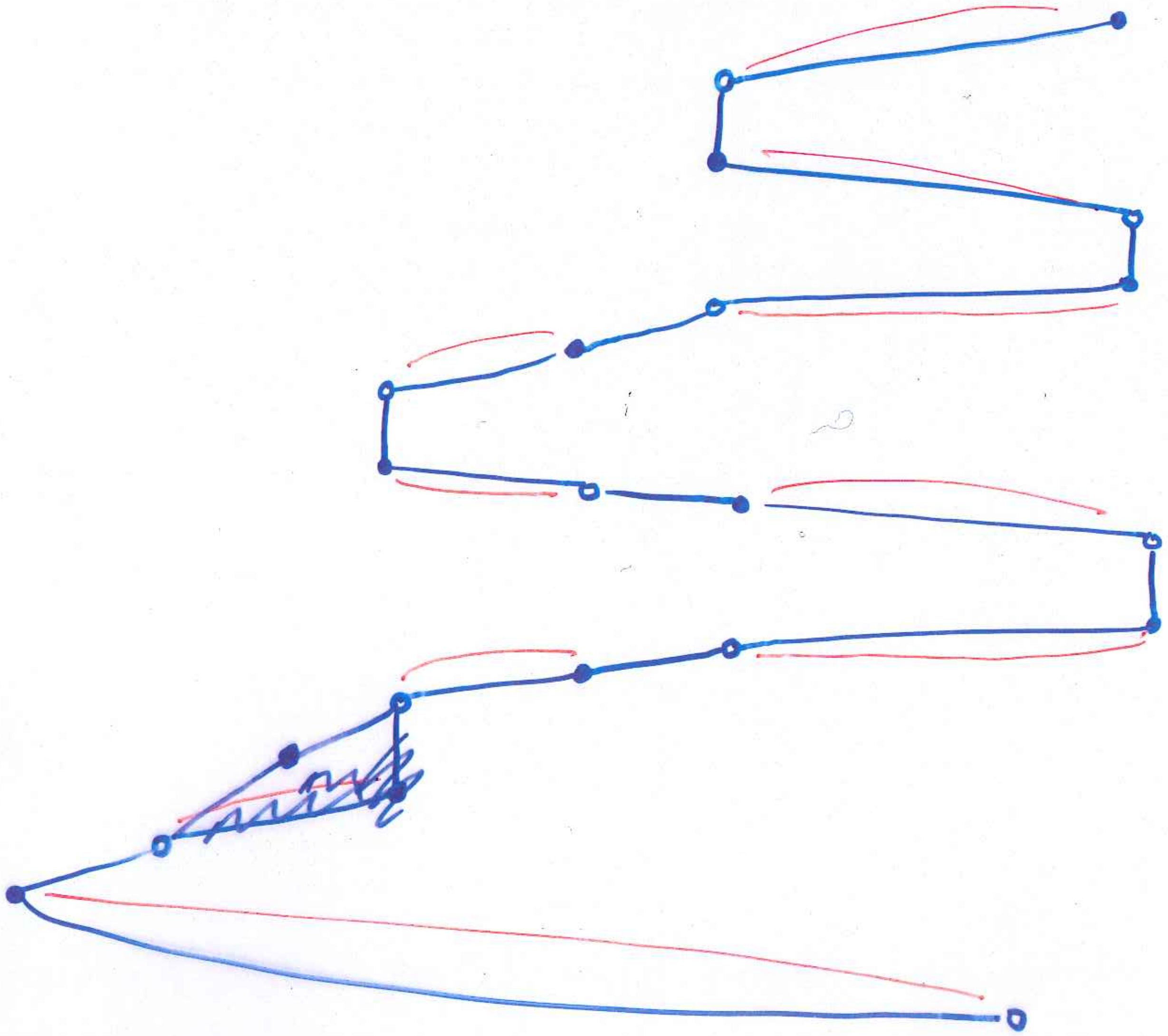
(Rerun)

$$\lambda f : A \Rightarrow A\ \lambda g : A \Rightarrow B\ \lambda a : A.\ \ g(f(f(a))) :$$

$$(A \Rightarrow A) \Rightarrow (A \Rightarrow B) \Rightarrow (A \Rightarrow B)$$

$$(A \Rightarrow A)_0^{\perp} \quad (A \Rightarrow B)_0^{\perp} \qquad A \Rightarrow B$$

$$(A \Rightarrow A)_1^{\perp}$$



PCF (SKETCH)

Simply typed λ-calculus
(with pairing if you wish)
on base types : bool, int .

Constants: true , false : bool
n : int ($n \in \mathbb{N}$)

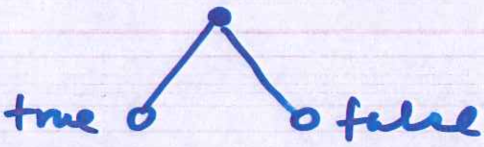
Arithmetic: succ , pred : int $\Rightarrow$ int
zero? : int $\Rightarrow$ bool

Conditional: if b then v else v' :
bool $\times A \times A \Rightarrow A$

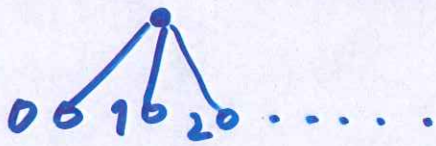
Fixed points: $Y : (A \Rightarrow A) \Rightarrow A$

INTERPRETATION OF PCF TYPES

bool



nat

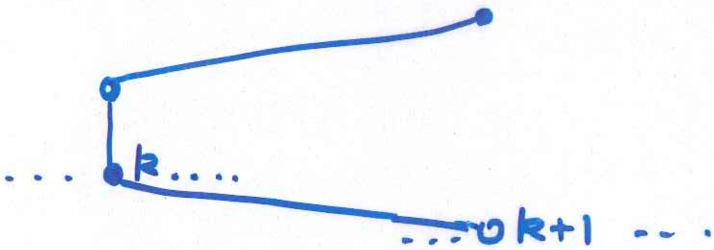


Higher types given by $\times$, $\Rightarrow$ as usual.

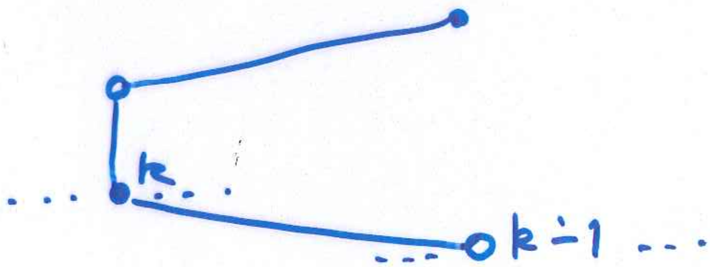
INTERPRETATION OF ARITHMETIC

(Just show maximal plays)

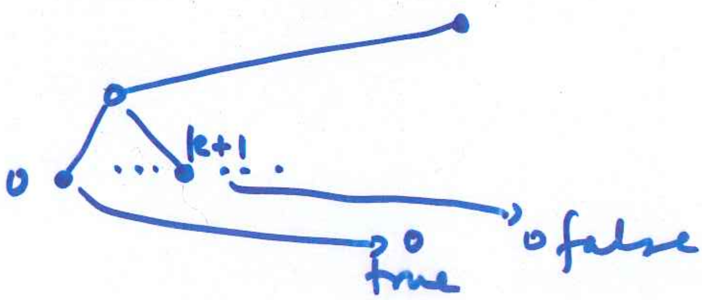
succ $\quad$ nat $\Rightarrow$ nat



pred $\quad$ nat $\Rightarrow$ nat



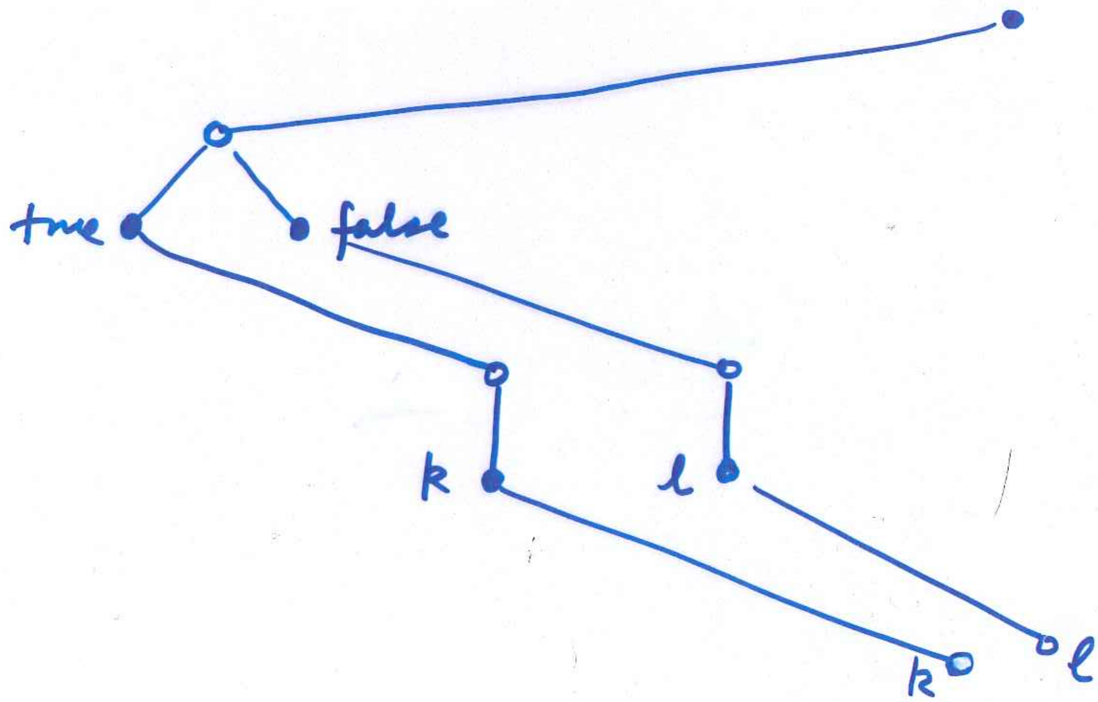
zero? $\quad$ nat $\Rightarrow$ bool



INTERPRETATION OF CONDITIONAL

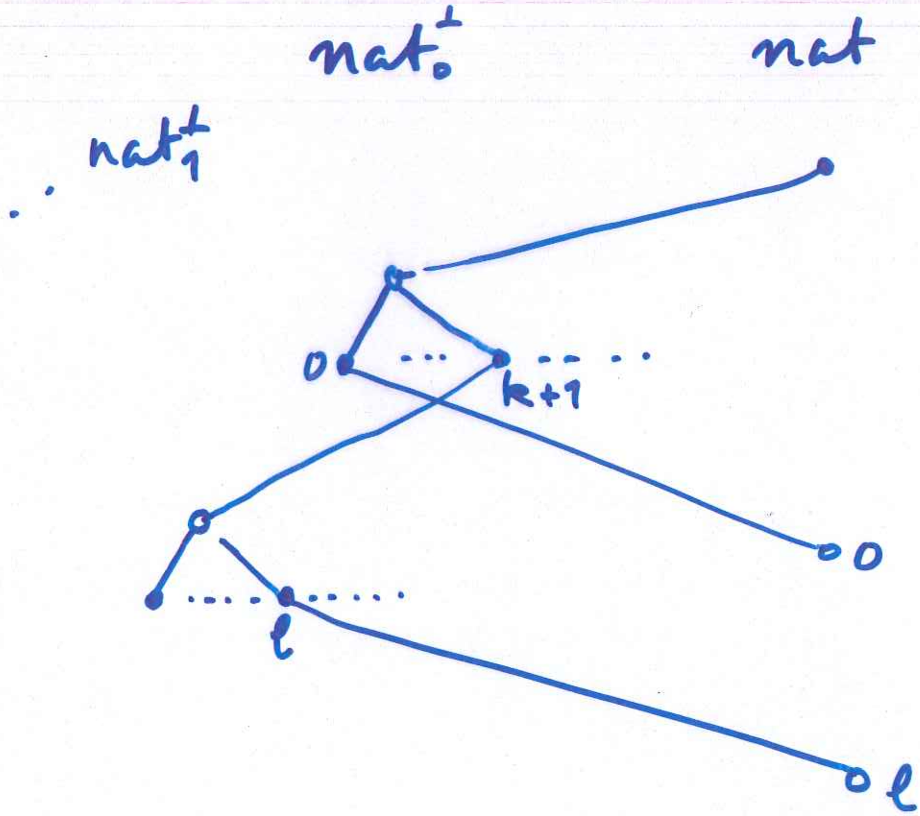
(Just show maximal plays)

$$\begin{pmatrix} \text{if } b \\ \text{then } v \\ \text{else } v' \end{pmatrix}^{nat} : bool \times nat \times nat \Rightarrow nat$$



EXAMPLES OF DEFINABLE STRATEGIES

① If $x = 0$ then 0 else x [Non-linear!]

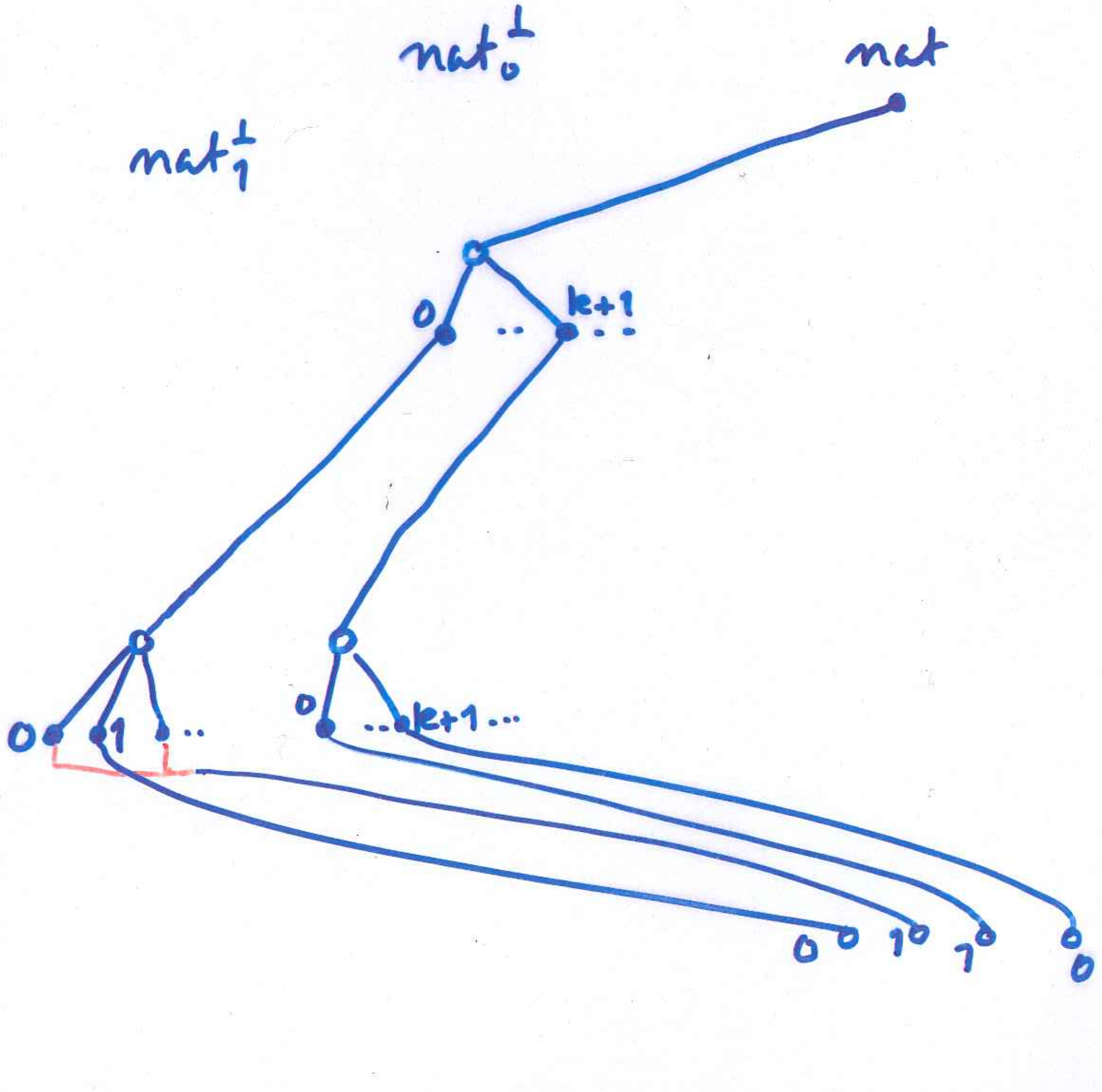


② If $x = 0$ then 0 else succ(x)

③ If $x = 0$ then (if $x = 0$ then 0 else 1) else 1

ANOTHER DEFINABLE STRATEGY

if $x=0$ then (if $x=1$ then 0 else 1)
else (if $x=0$ then 1 else 0)

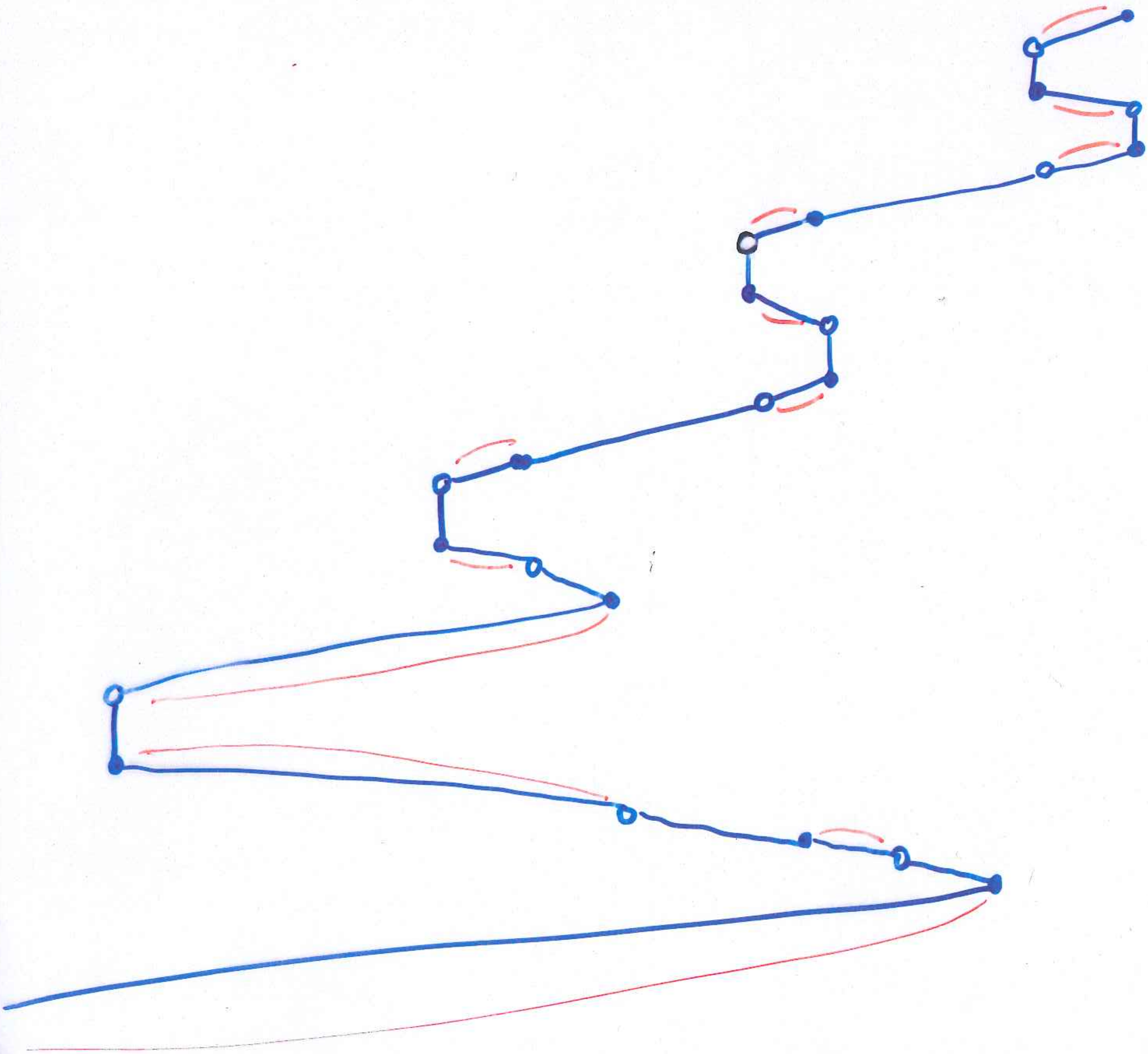


INTERPRETATION OF THE FIXED POINT Y.

$$Y : (A \Rightarrow A) \Rightarrow A$$

$$(A \Rightarrow)_3^{\perp} \qquad (A \Rightarrow A)_2^{\perp} \qquad (A \Rightarrow A)_1^{\perp} \qquad (A \Rightarrow A)_0^{\perp}$$

$$\ldots A_3^{\perp})\,(A_{20}\ A_{21} \ldots A_2^{\perp})\,(A_{10}\ A_{11} \ldots A_1^{\perp})\,(A_{00}\ A_{01} \ldots A_0^{\perp})\ A$$



PROPERTIES OF Y.

All hold at the level of strategies!!

FPP $$Y(f) = f(Y(f)) \qquad f : A \Rightarrow A$$

Dinaturality

$$Y(g \circ f) = g(Y(f \circ g)) \qquad \begin{array}{l} f : A \Rightarrow B \\ g : B \Rightarrow A \end{array}$$

Diagonal property $\qquad f : A \times A \Rightarrow A$

$$Y(\lambda a.\, f(a,a)) = Y(\lambda a.\, Y(\lambda a'.\, f(a,a')))$$

Plotkin naturality

$$h(Y(f)) = Y(g)$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & A \arrow[d, "h"] \\ B \arrow[r, "g"'] & B \end{tikzcd}$$

PROOF OF FPP

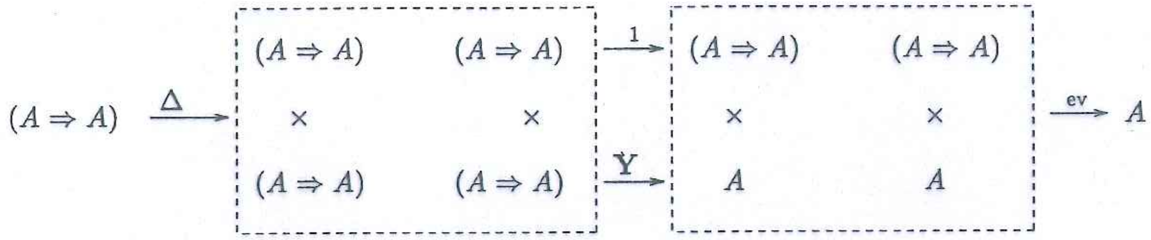


Figure 6: Composition of maps $\Delta; \mathbf{1} \times \mathbf{Y}; \mathrm{ev}$.

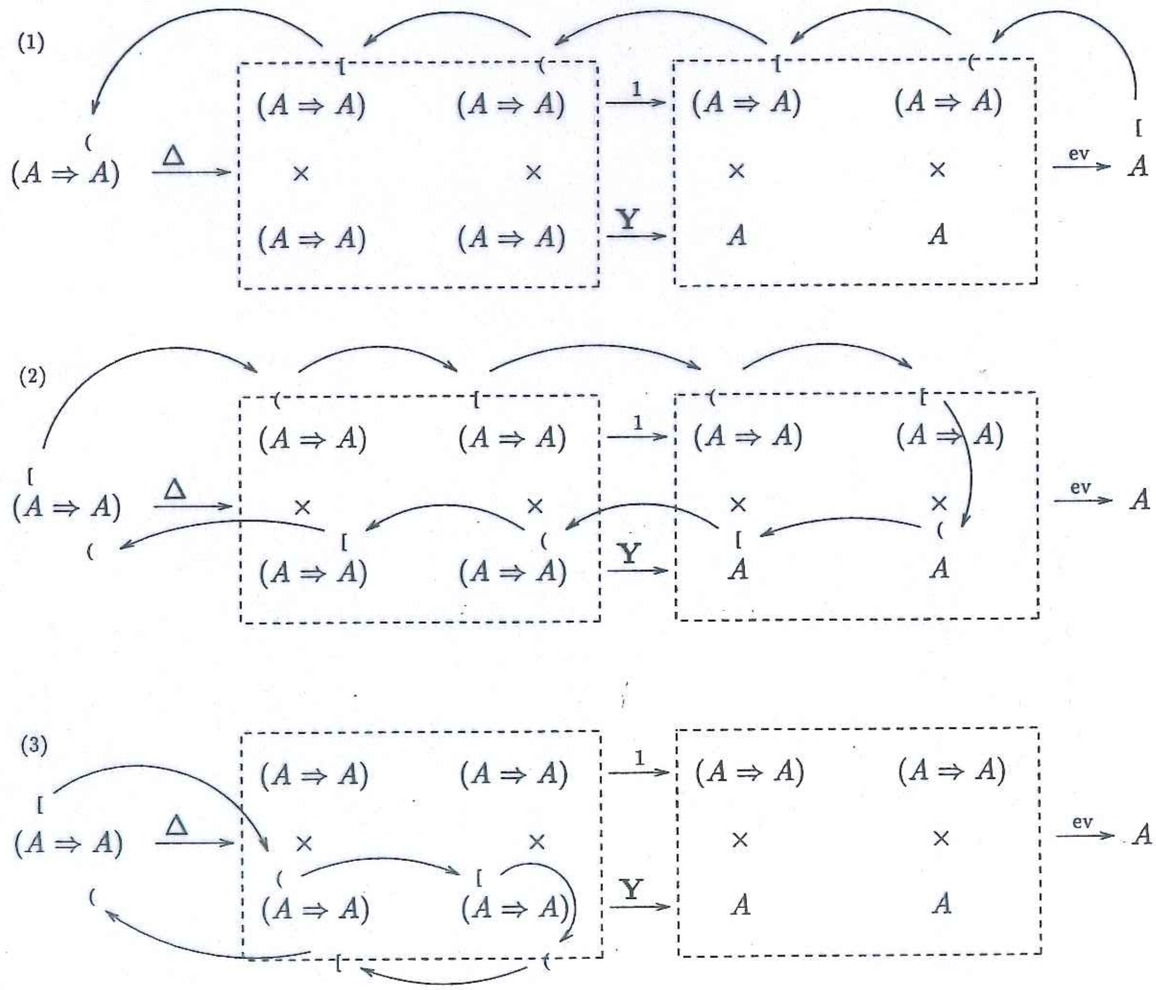
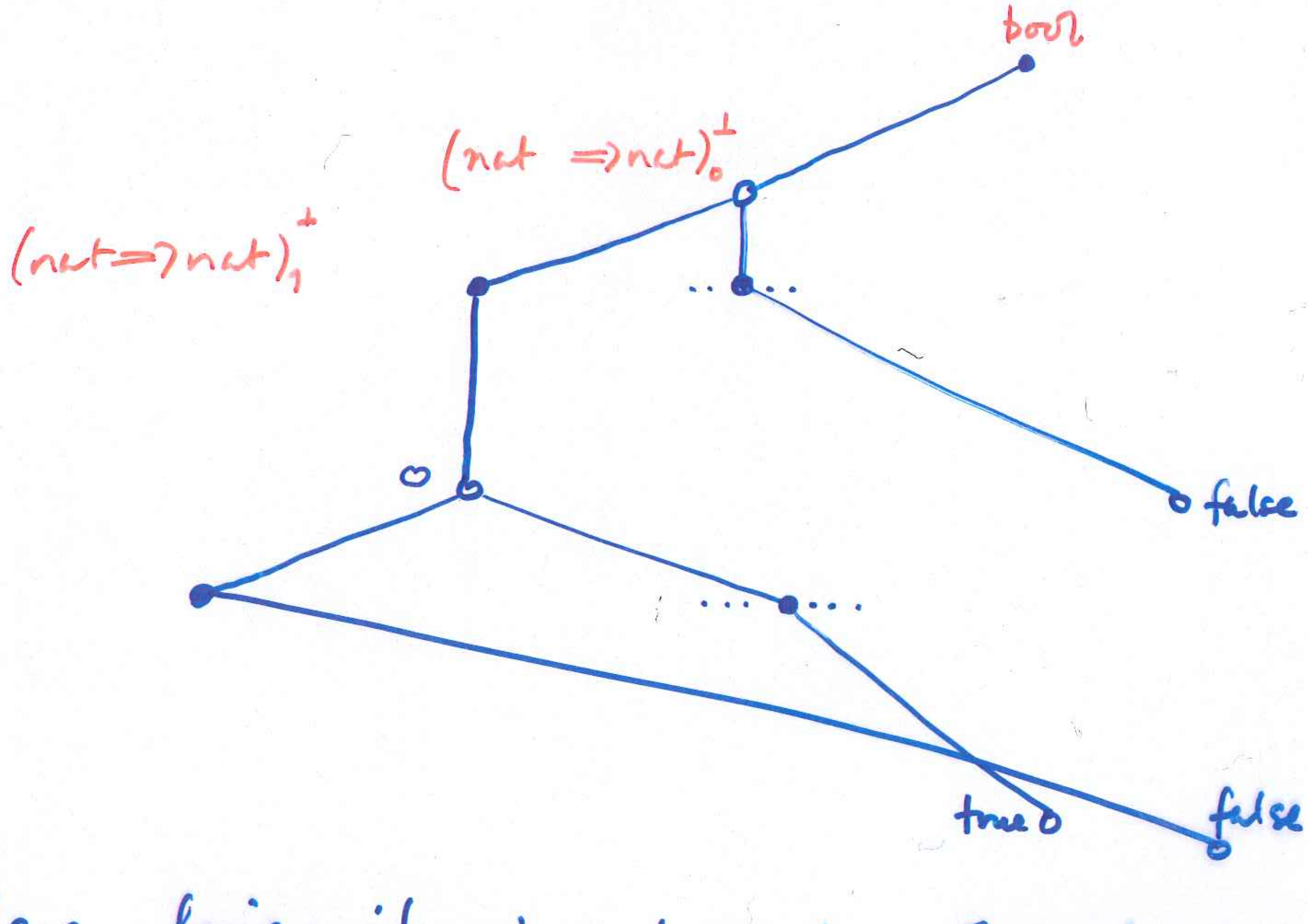


Figure 7: Copying paths.

TEST FOR LINEARITY AT 0.

(Just show maximal plays.)

lin 0? : $(nat \Rightarrow nat) \Rightarrow bool$

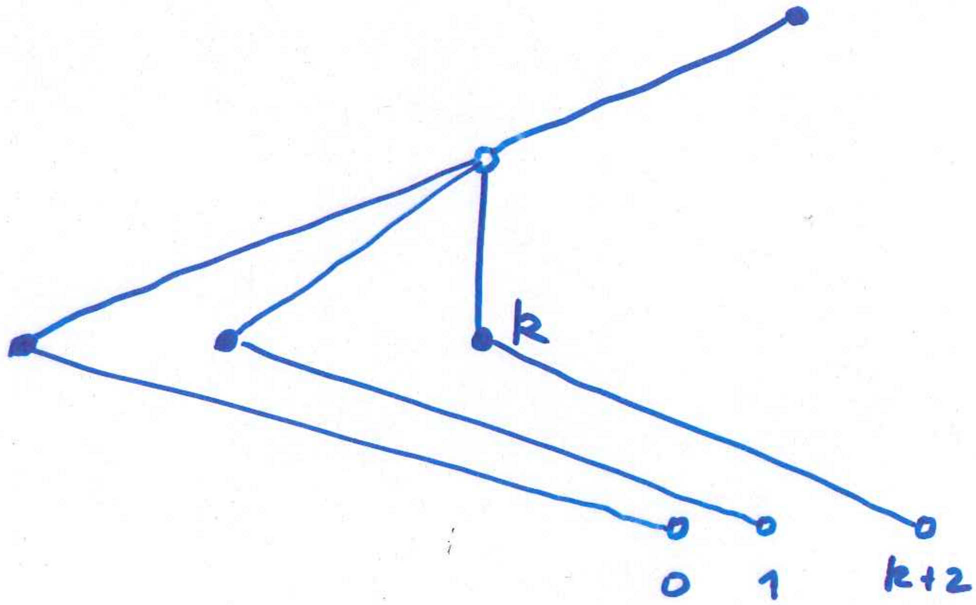


Here linearity is a property of strategies not of extensional functions.
This is a definitely intensional algorithm

CATCH

(Just show maximal plays)

catch : $(nat \times nat \Rightarrow nat) \Rightarrow nat$



This strategy acts extensionally but is not PCF definable.

WARNING

Cartwright, Curien & Felleisen:

The <u>extensional reduction</u> of this model is fully adequate for PCF + Catch

Not the 'observational quotient' as stated !!

GAMES & WINNING STRATEGIES

(Recap)

Each ∞ play p of A

is designated

Not either a win for Player: $|p| = W$

Constructive or a loss for Player: $|p| = L$

Normal play convention:

for finite plays, $|\langle\rangle| = W$,

p ending in $\circ$, so O to move has $|p| = W$,

p ending in $\bullet$, so P to move has $|p| = L$.

TENSOR & LINEAR FUNCTION SPACE

(Recap)

P wins $A \otimes B$ $\equiv$

P wins A and P wins B

P wins $A \multimap B$ $\equiv$

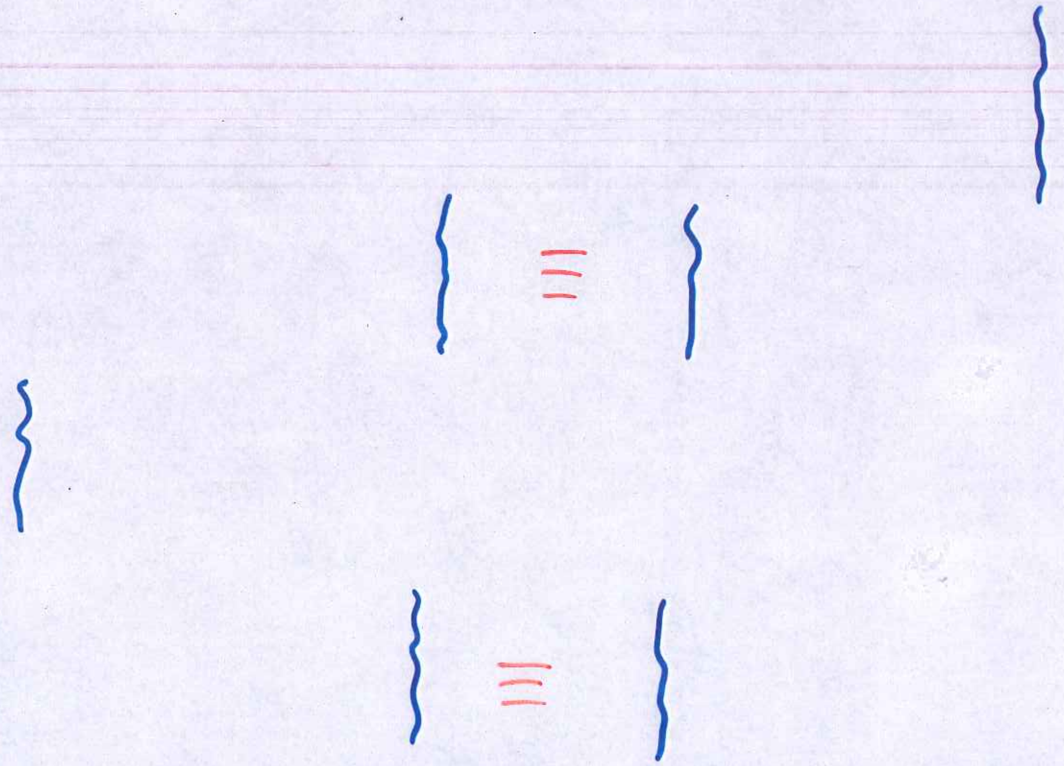
P wins A implies P wins B

[P wins $A^\perp$ or P wins B]

(True also for finite positions.)

COMPOSITION

$A \xrightarrow{\sigma} B \qquad B \xrightarrow{\tau} C$



"If P wins A then as σ winning
P wins B so as τ winning
P wins C."

EXPONENTIAL COMONAD

P wins $!A$ $\equiv$

P wins all the versions

$A_0, A_1, ..$ of A

Intuitively add

'which are played'

but formally P wins all

the versions which O does

~~not start~~.

- If O plays only in A_0 then just need to win that.

- If O keeps on opening new games (+ finishes none) then just need to keep on replying.

CATEGORIES OF WIN-GAMES

As for games with
partial strategies
we get

- a SMCC of games (to win) + (linear) winning strategies
- a CCC of games (to win) + (general) winning strategies.

Look at this as a model
the proof theory of $\wedge, \Rightarrow$ logic.

WEAK COMPLETENESS?

We have no notion of uniformity 'in play' so ask:

<u>if</u> for every $\vec{A} = A_1, \ldots, A_n$ there is a map

$$\Phi(\vec{A}) \longrightarrow \Psi(\vec{A}) \text{ in}$$

(general) games to win

<u>then</u> is $\Phi \vdash \Psi$ provable in intuitionistic logic $(\wedge, \Rightarrow)$.

FACT (trivial) if the $\vec{A}$ are all determined games then there is a map $\Phi(\vec{A}) \longrightarrow \Psi(\vec{A})$ whenever $\Phi \vdash \Psi$ classically.

NON-DETERMINED GAMES

Blass's Method : Fund. Math. 1972

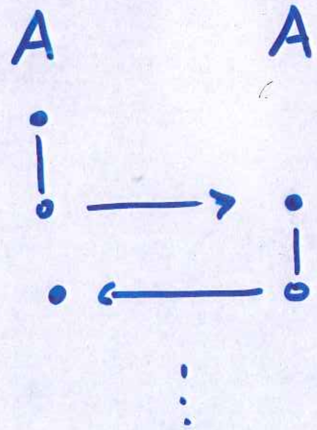
Game tree $T_A = \{ p \in \mathbb{N}^* \mid p_0 < p_1 < p_2 \cdots \}$.

$\text{Even}(p) = [p_0, p_1) \cup [p_2, p_3) \cup \cdots$

$$|p| = W \equiv \text{Even}(p) \in \mathcal{U}$$

where $\mathcal{U}$ fixed non-principal u.f. on $\mathbb{N}$.

- O wins $A \otimes A$



(so P doesn't win A)

- O wins $\bullet A^\perp \otimes \bullet A^\perp$ similarly

(so P doesn't win $\bullet A^\perp$
so "P doesn't win $A^\perp$"
so O doesn't win A)

So A is not determined

PLAUSIBLE CONJECTURE?

Assuming the Axiom of Choice (?)
the category of (general)
games to win is
weakly complete for the
$(\wedge, \Rightarrow)$-fragment of intuitionistic
logic.

Questions & Answers

All moves classified as either question or answer

+ this <u>not</u> changed by $A \mapsto A^{\perp}$.

Notation

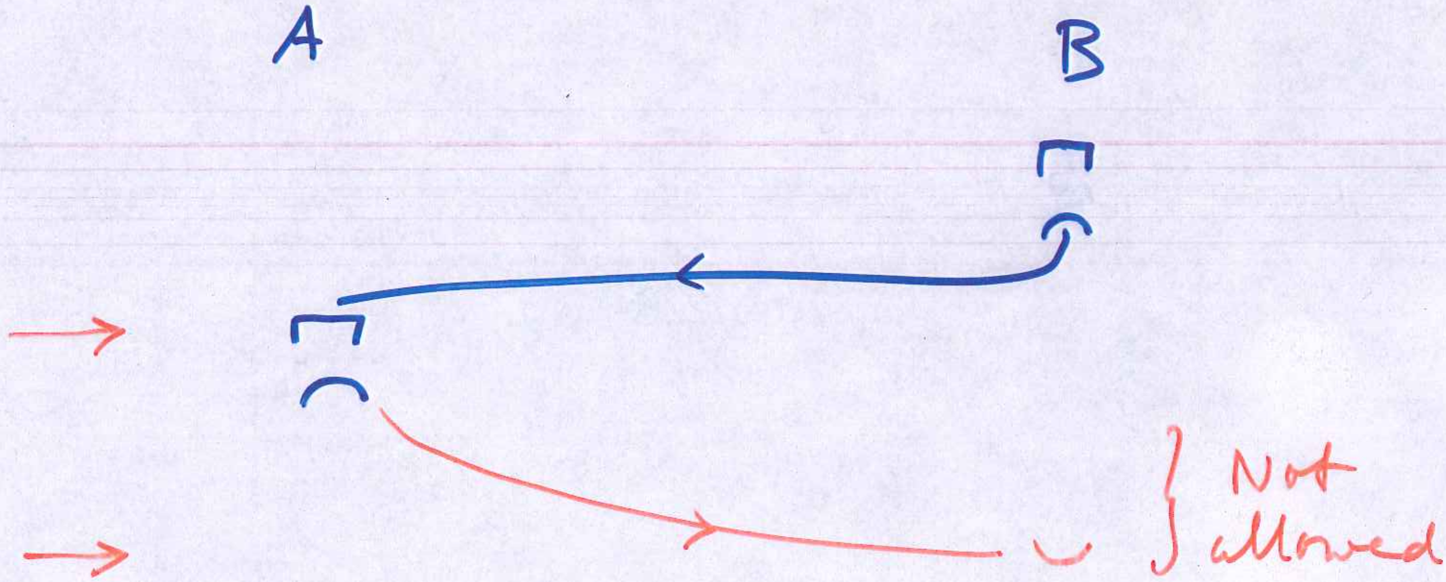
O's	question	[
P's	answer	]
P's	question	(
O's	answer	)

Braketing convention: plays look like [( ) ([ ] [([ ] [( )]) (

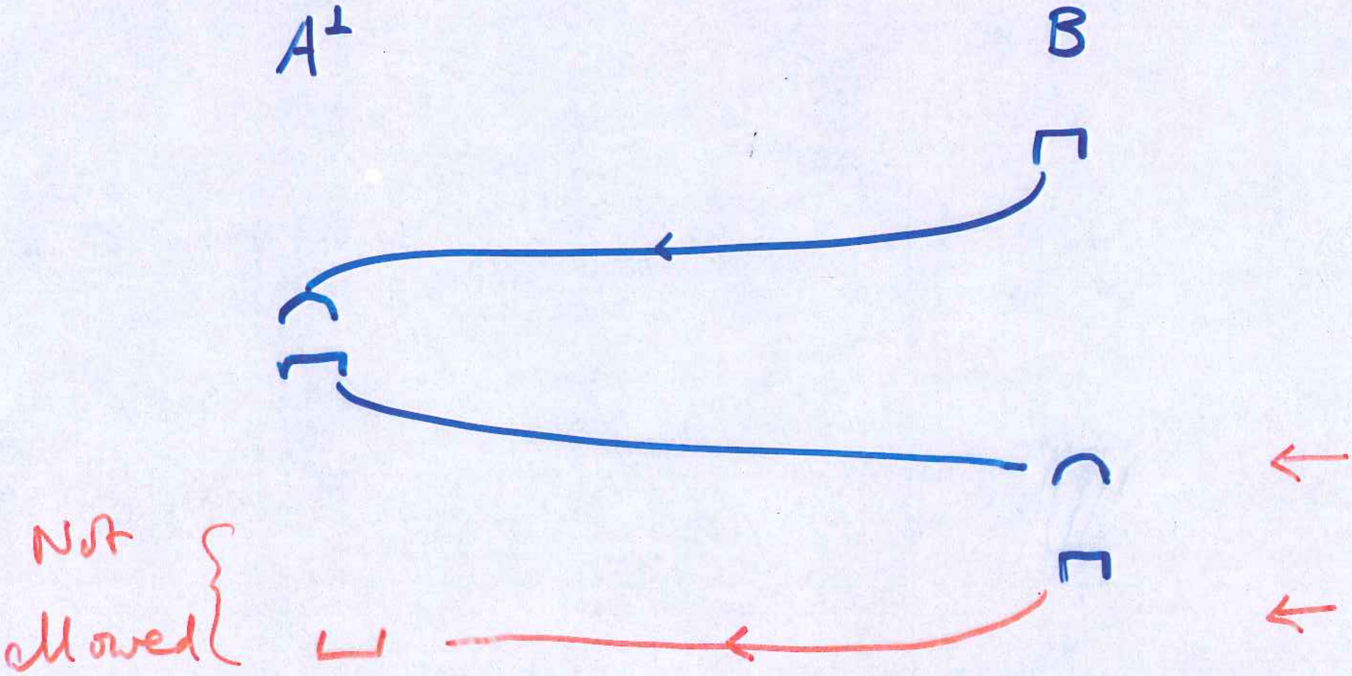
Hygiene: 'before closing a channel close all subsidiary ones'

Qs & As in $\otimes$ $\multimap$

$A \otimes B$



$A \multimap B$



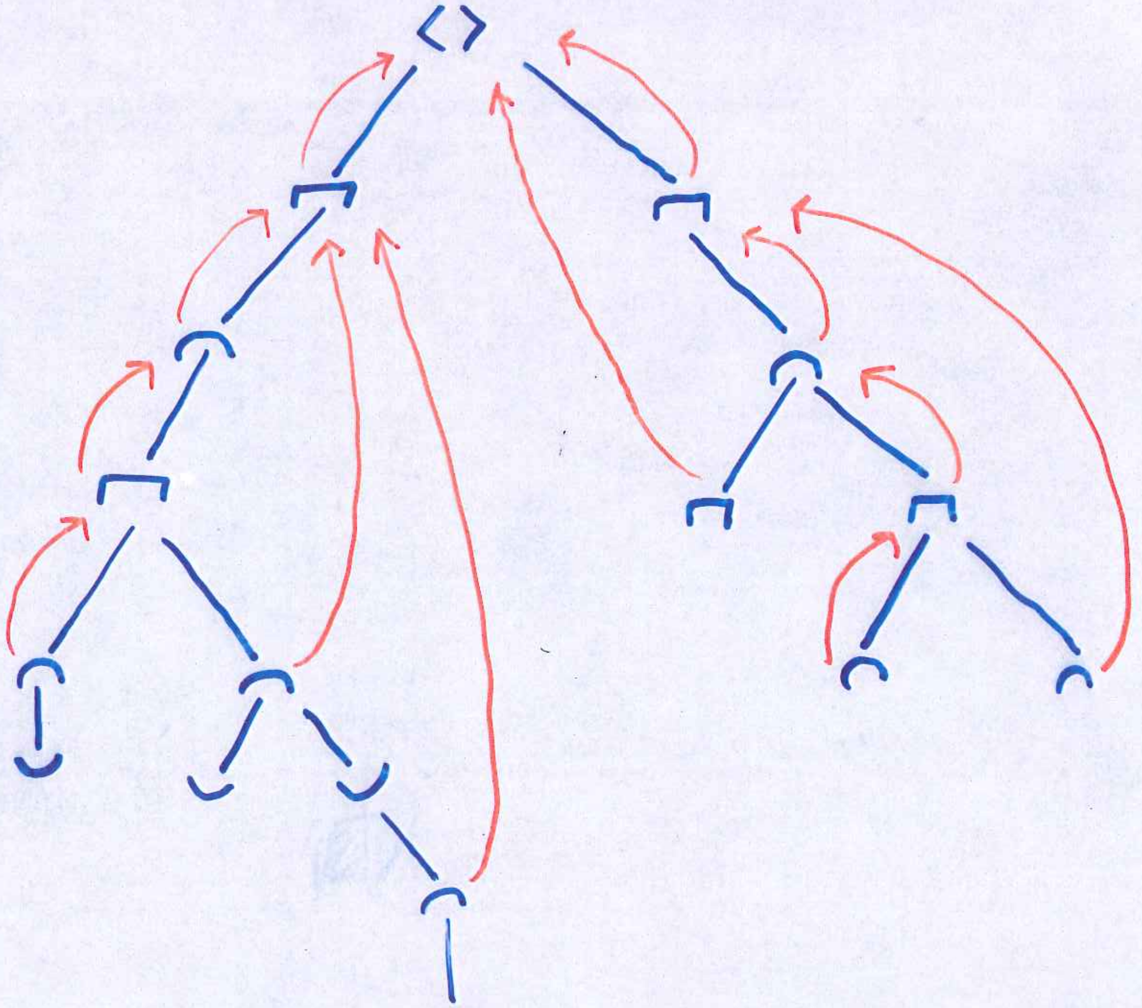
Justification

(of questions by questions only)

Question occurences are

EITHER fundamental questions

OR justified by an earlier unanswered question instance.



VIEWS

For a position p in which O has just played we have the P-view $\ulcorner p \urcorner$ of p:

$$\ulcorner [\urcorner = [$$

$$\ulcorner q \,.\, (\,.\, r \,.\, [\urcorner = \ulcorner q \urcorner \,.\, (\,.\, [$$

$$\ulcorner q \,.\, (\,.\, r \,.\,) \urcorner = \ulcorner q \urcorner \,.\, (\,)$$

Not as in notes

Exercise.

Player just considers the situation "as if Opponent at once played his moves" : i.e. systematically all moves between the ends of justification pointers are deleted.

Typical P-view:

$$[\;(\;)\;(\;)\;(\;[\;(\;)\;(\;)\;(\;[$$

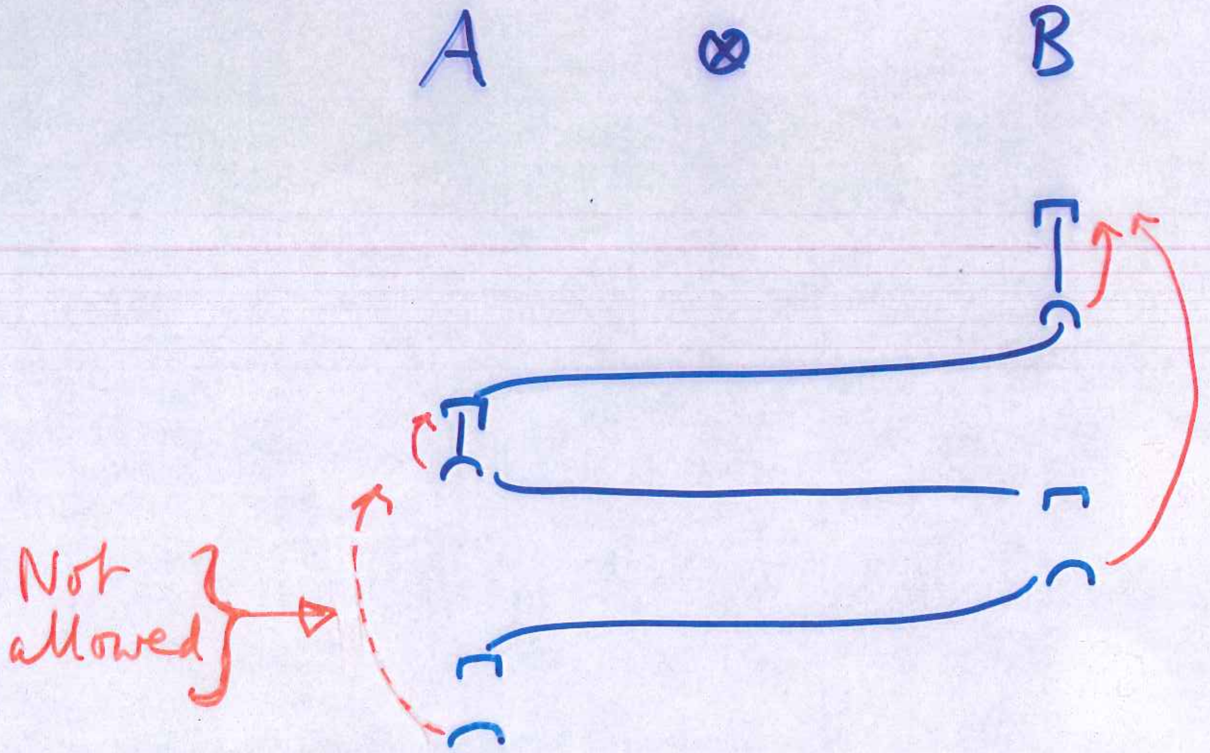
LEGAL POSITIONS

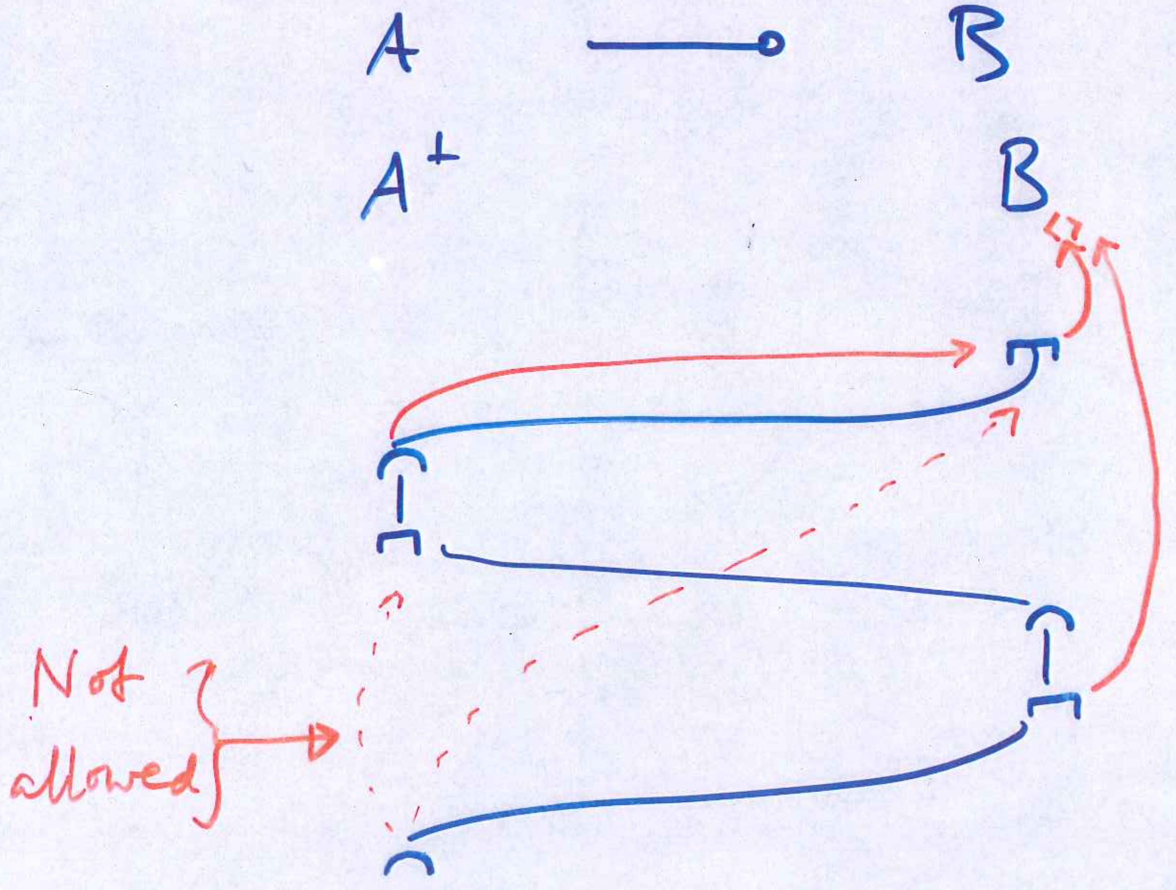
BUT justifying moves may disappear in the P-view.

A position / play is legal if every justifying move (question) is in the $\left\{\begin{array}{l}\text{P-view}\\ \text{O-view}\end{array}\right\}$ of the position at the point that $\left\{\begin{array}{l}\text{Player}\\ \text{Opponent}\end{array}\right\}$ is about to play.

A game with Qs and As, justification + legal positions is a <u>dialogue game</u>.

LEGALITY & $\otimes$ $\multimap$





INNOCENT STRATEGIES IN DIALOGUE GAMES

A strategy in dialogue game is <u>innocent</u> if and only if

" the reply of σ to p
= the reply of σ to q
whenever $\ulcorner p \urcorner = \ulcorner q \urcorner$ "

where relevant

So σ given by map from P-views to moves.

An innocent strategy is <u>finite</u> if and only if the map

$$\text{P-views} \longrightarrow \text{Moves}$$

is finite.

CATEGORIES OF DIALOGUE GAMES

As before there are

- SMCC s of dialogue games + innocent strategies — Linear
- CCC s of dialogue games & innocent strategies — general

Concentrate on games to win.

GAMES FOR FINITARY LOGIC

Definition: [For Dialogue Games]

The game A is balanced
if and only if

- there are no justification cycles in A

- whenever p is an infinite play in A and Player only often asks questions justified by an instance $[$ in A then Player loses

- + similarly for Opponent.

The game A is finitely generated
if and only if
the number of questions and answers is finite.

GAMES FOR FINITARY LOGIC CTD.

Balanced, finitely generated dialogue games form a mbccc of the ccc of dialogue games

The main point:

$$\frac{A\ ,\ B\ \text{ balanced}}{A \Rightarrow B\ \text{ balanced}}$$

A COMPACTNESS THEOREM FOR PROOFS

Consider dialogue games to win and amongst them the (sub c.c.c. of) balanced finitely generated games } 'finitary logic'

THEOREM

All winning strategies are finite

(Only dealing with innocent strategies.)

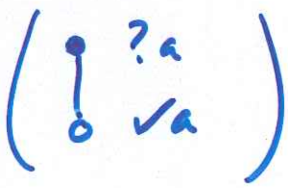
ARGUMENTS

Fix a set of propositional constants
$\{a, b, c, \ldots\}$

An <u>argument</u> is a dialogue whose answers are $a, b, c, \ldots$ and whose questions are associated with a unique answer.

—

Interpret logic by

$a \longmapsto$ 

'GOOD' STRATEGIES

A play p is good for player iff whenever player confesses a opponent has already done so.

A strategy is good iff all plays according to it are good.

N.B. This is a clear feature of uniform strategies.

WEAK COMPLETENESS

Take the category of arguments and good winning strategies*.

THEOREM

For the canonical interpretation, if there is $\sigma : \Phi \to \Psi$ then $\Phi \vdash \Psi$ in intuitionistic logic.

* innocent of course!

PROOF OF WEAK COMPLETENESS

Show:

If $\sigma : \Phi \Rightarrow \Psi$

is winning (innocent) and

bad for $\vec{a}$, then

$$\vec{a}, \Phi \vdash \Psi$$

in intuitionistic logic

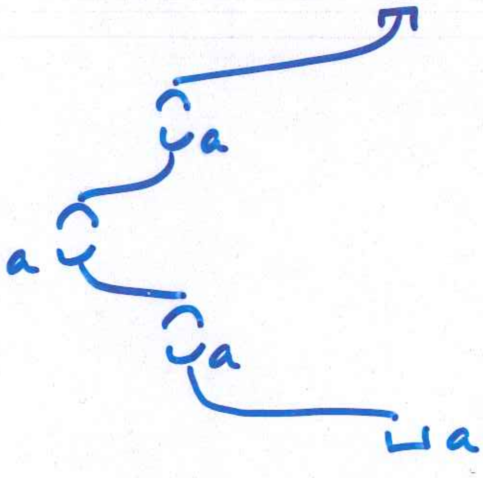
by induction on the

size of σ (finite).

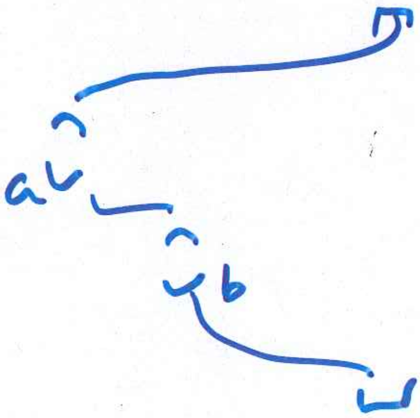
HOW STRONG?

Not very!

$$a \wedge a \Rightarrow a$$



$$a \wedge b \Rightarrow b$$



PROBLEM OF UNIFORMITY IN CLASSICAL PROOF THEORY

- Girard symmetry argument

- In a ccc an initial object is strict + so each object of form

$$A \Rightarrow 0$$

is a subterminator.

Perhaps interpret $\perp$ not as 0 but just weakly initial (+ require $A \lhd \neg\neg A$ say)?

(But not satisfactory cf. Gentzen interpretation.)

Uniformity?

COQUAND ON GENTZEN'S FIRST PROOF

$\Gamma \vdash A$

O's steps

- replace free vble by numeral
- replace $\forall x\, A$ by $A(n)$
- replace $A \wedge B$ by A or by B

— replace $\Gamma \vdash C \Rightarrow \bot$ by $\Gamma, C \vdash \bot$.

P's steps

- replace $\Gamma, \forall x\, A \vdash \bot$ by
  $\Gamma, \forall x\, A, A(n) \vdash \bot$ (or)
- replace $\Gamma, A \wedge B \vdash \bot$ by
  $\Gamma, A \wedge B, A \vdash \bot$ or $\Gamma, A \wedge B, B \vdash \bot$ (or)

— replace $\Gamma, C \Rightarrow \bot \vdash \bot$ by

$\Gamma \vdash C$

INTERPRETATION IN WIN-GAMES

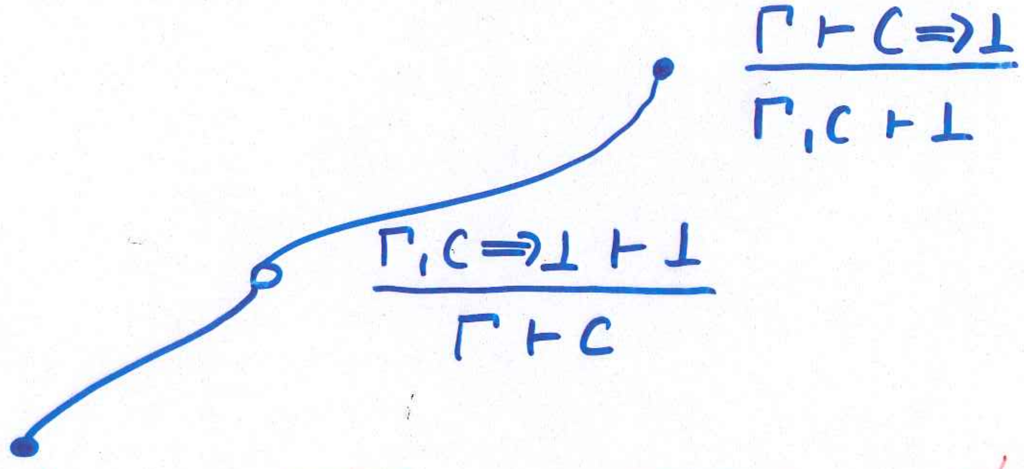
GAME is $\prod_{\vec{n}\in\mathbb{N}^k} \Gamma(\vec{n}) \Longrightarrow A(\vec{n})$

Interpret a true closed atomic by

$$(\ ) = 1;$$

a false closed atomic by $(\bullet) = S$.

No back tracking so an essentially "linear play"



(Effectively a definition of truth)

GENTZEN: (By 'elementary means')

If there is a proof then there is a winning strategy