POLARISED LINEAR LOGIC

Negative formulae

$$N = X^{\perp} \mid N ⅋ N \mid N \& N \mid \bot \mid \top \mid ?P$$

Positive formulae

$$P = X \mid P \otimes P \mid P \oplus P \mid 1 \mid 0 \mid !N$$

Rules in terms of one sided sequents

$$\vdash \Gamma$$

with the stipulation that

there is at most one positive formula in Γ.

Some Rules

$$\frac{\vdash \Gamma, N, M}{\vdash \Gamma, N ⅋ M} \qquad \frac{\vdash \Gamma, P \qquad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P \otimes Q}$$

$$\frac{\vdash \mathcal{N}, N}{\vdash \mathcal{N}, !N} \qquad \frac{\vdash \Gamma, P}{\vdash \Gamma, ?P} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, N} \qquad \frac{\vdash \Gamma, N, N}{\vdash \Gamma, N}$$

CUT $$\frac{\vdash \Gamma, N \qquad \vdash N^{\perp}, \Delta}{\vdash \Gamma, \Delta}$$

NATURAL MODEL

(Regnier)

Take $\mathbb{L}$ a model for classical linear logic. Then let

Negative formulae be ⅋ c.monoids

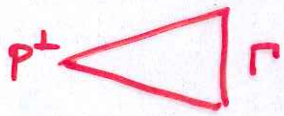
Positive formulae be $\otimes$ c.comonoids

and take relevant sequents

NEGATIVE READING

Read $\vdash \Gamma, P$

as $P^{\perp} \vdash \Gamma$

i.e. 

and $\vdash N$

as 

i.e. an opmulticategory acting on a set on the (right/left).

We move to a positive reading i.e. get a multicategory on the positive formulae.

VERSIONS OF POLARISED LOGIC

Issues (positive reading)

With additives $\oplus$ or Without additives

Exponentials or Lifts
(so $\otimes$ a product) ($\otimes$ arbitrary)

	$\oplus$	no $\oplus$
Exponentials	control cocontrol world	$\otimes \neg$ world
Lifts		

Simplest so study that first.

[Rest will lift along same lines.)

CATEGORICAL SYNTAX

We have $\mathbb{P}$ a symmetric monoidal category, with

Pos a set on which $\mathbb{P}$ acts on the right

i.e. $Pos : \mathbb{P}^{op} \longrightarrow Set$

an operation $\neg$ on objects

elements $\pi_A \in Pos(\neg A \otimes A)$

operations $\lambda_B : Pos(A \otimes B) \longrightarrow \mathbb{P}(A, \neg B)$

natural in A,

satisfying $\lambda_A(\pi_A) = id_{\neg A}$

for $\phi \in Pos(A \otimes B)$ $(\lambda_B \phi \otimes B)^* \pi_B = \phi$

Naturality is

$$\lambda_B\left((u \otimes B)^* \phi\right) = \lambda_B \phi . u$$

for $\phi \in Pos(A \otimes B)$ and $u : A' \longrightarrow A$.

NATURAL DEDUCTION VERSION

Negative types

$$T = X \mid T \otimes T \mid I \mid \neg T$$

Positive type $\qquad P$

Usual $\otimes$-calculus

$$\frac{}{* : I} \qquad \frac{\Gamma \vdash t : I \qquad \Delta \vdash a : A}{\Gamma, \Delta \vdash \text{let } t \text{ be } * \text{ in } a : A}$$

$$\frac{\Gamma \vdash a : A \qquad \Delta \vdash b : B}{\Gamma, \Delta \vdash a \otimes b : A \otimes B} \qquad \frac{\Gamma, x : A, y : B \vdash c : C \qquad \Delta \vdash s : A \otimes B}{\Gamma, \Delta \vdash \text{let } s \text{ be } x \otimes y \text{ in } c : C}$$

Response rules

$$\frac{\Gamma \vdash a : A}{\Gamma, x : \neg A \vdash xa : P} \qquad \frac{\Gamma, x : A \vdash \phi : P}{\Gamma \vdash \lambda x . \phi : \neg A}$$

ALGEBRAIC FORM

OF THE CATEGORICAL COMBINATORS

(i.e. we write the action on the right)

Naturality

$$\lambda_B(\phi . u \otimes B) = \lambda_B \phi . u$$

β-rule

$$\pi_B . (\lambda_B \phi \otimes B) = \phi$$

η-rule

$$\lambda_A \pi_A = id_{\neg A}$$

A NATURAL ISOMORPHISM

We have correspondences

$$\begin{aligned} \mathrm{Pos}(A \otimes B) &\longrightarrow \mathbb{P}(A, \neg B) \\ \phi &\longmapsto \lambda_B \phi \end{aligned}$$

$$\begin{aligned} \mathbb{P}(A, \neg B) &\longrightarrow \mathrm{Pos}(A \otimes B) \\ f &\longmapsto \pi_B . (f \otimes B) \end{aligned}$$

The composites

$$\phi \longmapsto \pi_B . (\lambda_B \phi \otimes B) = \phi \quad \text{by } \beta\text{-rule}$$

$$\begin{aligned} f \longmapsto \lambda_B (\pi_B . (f \otimes B)) &= \lambda_B \pi_B . f && \text{by naturality} \\ &= id_{\neg B} . f && \text{by } \eta\text{-rule} \\ &= f \end{aligned}$$

So

$$\mathrm{Pos}(A \otimes B) \cong \mathbb{P}(A, \neg B)$$

(naturally in A).

RESPONSE CATEGORIES

These are symmetric monoidal categories $\mathbb{C}$ equipped with an object R and special function spaces $A \multimap R$ with natural isomorphisms

$$\mathbb{C}(A \otimes B, R) \cong \mathbb{C}(A, B \multimap R).$$

ALTERNATIVE PRESENTATION

Symmetric monoidal $\mathbb{C}$ with a strong self-duality $N : \mathbb{C} \longrightarrow \mathbb{C}^{op}$ on the right.

$\mathbb{P}$ AS RESPONSE CATEGORY

We have

$$\mathbb{P}(A, \neg B) \cong Pos(A \otimes B)$$

$$\cong Pos(A \otimes B \otimes I) \cong \mathbb{P}(A \otimes B, \neg I)$$

so set

$$R = \neg I$$

and

$$B \multimap R \cong \neg B .$$

TENSOR - NOT CATEGORIES

For this talk, these are the premonoidal categories which occur as the Kleisli categories for the

$$((\ )\multimap R)\multimap R \;=\; \neg\neg(\ )$$

monad on a response category.
(i.e. drop the cartesian condition)

We have

$$\mathbb{C}_{\neg\neg}(A, B) = \mathbb{C}(A, \neg\neg B) \cong \mathbb{C}(\neg B, \neg A)$$

with premonoidal structure $\otimes$.

Closed Multicategories

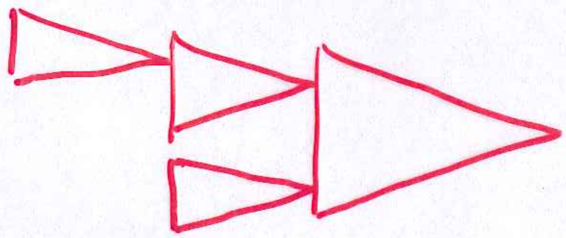
A multicategory $\mathbb{M}$ has

objects A, B, C

multimaps

$$f : A_1 \cdots A_n \longrightarrow B$$

with obvious identities and composition laws, e.g.



(We are interested in symmetric $\mathbb{M}$.)

$\mathbb{M}$ is closed if it is equipped with function spaces $[A, B]$ and appropriately natural isomorphisms

$$\mathbb{M}(A_1, \ldots, A_n, B ; C) \cong \mathbb{M}(A_1, \ldots, A_n ; [B, C])$$

LAMBDA CALCULUS TRANSLATION

Let $\mathbb{D}$ be the opposite of our $\otimes\neg$ category $\mathbb{C}_{\neg\neg}$; so we have

$$\mathbb{D}(A, B) = \mathbb{C}_{\neg\neg}(B, A) \cong \mathbb{C}(\neg A, \neg B)$$

The functor $\neg : \mathbb{D} \longrightarrow \mathbb{C}$ induces a multicategory structure on $\mathbb{D}$:

$$\mathbb{D}(A_1, \dots A_n ; B) \cong \mathbb{C}(\neg A_1 \otimes \dots \otimes \neg A_n , \neg B)$$

$\mathbb{D}$ is then a closed multicategory as

$$\mathbb{D}(A_1, \dots A_n, B ; C) \cong \mathbb{C}(\neg A_1 \otimes \dots \otimes \neg A_n \otimes \neg B, \neg C)$$

$$\cong \mathbb{C}(\neg A_1 \otimes \dots \otimes \neg A_n , \neg(\neg B \otimes C))$$

$$\cong \mathbb{D}(A_1, \dots A_n ; \neg B \otimes C)$$

so that the closed structure is

$$[B, C] = \neg B \otimes C$$

Note in particular that

$$[B, \neg C] = \neg B \otimes \neg C \cong [C, \neg B]$$

so that $\neg$ is an internal / strong / enriched self-duality on the right. Indeed

$$\neg B \cong [B, I]$$

so I is now the response object.

EMBEDDING THEOREMS

(1) Any small (symmetric) closed multicategory $\mathcal{K}$ embeds by

$$\mathcal{K} \longrightarrow \mathcal{L}$$

in a small (symmetric) monoidal category $\mathcal{L}$, where $\mathcal{K} \longrightarrow \mathcal{L}$ preserves the closed structure of $\mathcal{K}$.

HENCE

(2) Any small (symmetric) closed multicategory $\mathcal{K}$ embeds by

$$\mathcal{K} \longrightarrow \tilde{\mathcal{L}} = [\mathcal{L}^{op}, \mathrm{Set}]$$

in a (symmetric) monoidal closed category $\tilde{\mathcal{L}}$, where $\mathcal{K} \longrightarrow \tilde{\mathcal{L}}$ preserves the closed structure of $\mathcal{K}$.

ENRICHED ENDOMORPHISMS

Take $\mathcal{K}$ a closed multicategory: let $\mathcal{E}$ be the category of enriched endomorphisms of $\mathcal{K}$ i.e.

objects X where

$$(A,B) \longmapsto [A,B] \xrightarrow{X} [XA,XB]$$

maps $X \xrightarrow{\alpha} Y$ where

$$XA \xrightarrow[\alpha_A]{} YA$$

satisfying axioms.

1. $\mathcal{E}$ and so $\mathcal{E}^{op}$ monoidal under composition

2. $\mathcal{K} \longrightarrow \mathcal{E}^{op}$ by $A \longmapsto [A,-]$

3. YONEDA LEMMA

$$\mathcal{E}([A,-],X) \cong \mathcal{K}(\cdot\,; XA)$$

4. For $\mathcal{K}$ symmetric can cut down to symmetric monoidal $\mathcal{L} \subseteq \mathcal{E}^{op}$.

A NEW RESPONSE CATEGORY

Take $\mathbb{K}$ a (symmetric) closed multicategory with object S and so induced self-duality $[-, S]$ on the right.

Let $\mathbb{R}$ be the closure in $\tilde{\mathbb{K}}$ of the image of $\mathbb{K}$ under $\otimes$ and $- \multimap S$; so $\mathbb{R}$ is the generated response category.

WE HAVE

Response category $\mathbb{C}$

$\Downarrow$

Tensor-not category $\mathbb{C}_{\neg\neg}$

$\Downarrow$

Closed multicategory $\mathbb{D}$ with object I

$\Downarrow$

Response category $\mathbb{R}$ as above

syntactic description